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# ANALYTICAL GEOMETRY OF THREE DIMENSIONS

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# PREFACE

THIS book is based upon a short course of lectures to first-year Honours students who have just completed a course on Algebra approximating to that covered by Dr Aitken's *Determinants and Matrices* in these Texts, and who will have later in their curriculum a course of modern projective geometry. While the original lecture notes have been drastically revised so that the book may, one hopes, meet the needs also of students whose courses are differently arranged, it is still designed primarily for students at about the same stage as those to whom the lectures were addressed.

This explanation, as well as the smallness of the book, will account for various omissions. For instance, homographic correspondence is not introduced, largely on the ground that it plays its fundamental part in non-metrical geometry which the student will normally encounter at a later stage. On the other hand, it is hoped that the topics included will be found to be treated with a reasonable standard of rigour. In particular, care has been taken to frame the theory so that it does strictly apply to *real* space. This explains the avoidance of certain familiar short-cuts, which actually depend on jumping difficulties about reality conditions.

The book purports to be a "University Text." This presumably means that it will normally be used in conjunction with lectures or other personal instruction. It

would therefore seem merely foolish to include the sort of additional explanation, illustrated by "trivial" examples, which is more appropriately given by word of mouth. At any rate, the author has felt justified in assuming that the majority of readers will not be working entirely without supervision, and has allowed this assumption to influence his manner of presenting the subject.

I gratefully acknowledge the valuable suggestions I have received from the editors and from my colleague, Dr R. Cooper, and the careful cooperation I have received from the printers.

W. H. McCREA

BELFAST, *July* 1942

# CONTENTS

CHAPTER I

# COORDINATE SYSTEM: DIRECTIONS

## 1. Introductory

We are going to study by algebraic methods the geometry of three-dimensional real euclidean space, usually regarded as "ordinary" space. We adopt the elementary view of analytical geometry, according to which it is merely a matter of convenience to introduce the algebraic method as a tool for the solution of problems having a well-defined meaning apart from the algebra. However, we shall observe that our tool guides us to those problems with which it is best fitted to deal.

This elementary treatment is useful in applications to other parts of mathematics and to mathematical physics. But also, in geometry itself, the student can scarcely hope to appreciate modern abstract treatments without some such introduction. Moreover, the algebraic manipulation in abstract geometry is not essentially different. The form in which it is cast in this book is chosen partly to meet the requirements of the student wishing to pursue the subject further, and for his benefit a note on abstract geometry is given at the end.

The reader is assumed to be acquainted with elementary pure solid geometry, and with simple analytical geometry of two dimensions. The only other special mathematical equipment required is some knowledge of determinants and matrices; for this, reference will be made to Dr Aitken's *Determinants and Matrices* in these Texts (quoted as "Aitken").

The examples consist almost entirely of auxiliary results needed in the general development. In some cases proofs are indicated and should be completed by the reader; in others proofs should be supplied by the reader as he proceeds. In each section, examples are numbered consecutively, and subsequent reference made by giving the number of the section followed by that of the example. Formulæ are similarly treated, except that their numbers are put in brackets. The reader is urged to construct for himself numerical exercises; for "riders" he must consult larger textbooks and examination papers.

**Metrical geometry.** The present geometry is metrical, which means that results are expressed, directly or indirectly, in terms of *distance* and *angle*. *Distance* expresses a relationship between a pair of points; *angle* a relationship between a pair of directions. Both magnitudes have to be measured by comparison with selected standards. It may be remarked that, whereas there is in euclidean space a natural standard angle (the right angle being a convenient unit), the standard of length is quite arbitrary. However, we are here assuming the fundamental properties of these magnitudes, and our initial consideration in applying algebraic methods to the geometry is to find means first of labelling points and directions by algebraic symbols and then of expressing distances and angles in terms of these symbols. Such is the object of this chapter.

**Nomenclature.** We call three-dimensional euclidean space $\mathcal{E}$. *Line* will always mean straight line; any other sort of "line" will be called a *curve*. If $A$, $B$ are any two distinct points, then we use the following notation:

"The line $AB$," or simply $AB$, means the *whole line* containing $A$, $B$, as distinct from "the segment $AB$";

$\mathbf{AB}$ denotes the same line when *sense* is relevant and is taken from $A$ to $B$;

$|AB|$ denotes the *length* of the segment $AB$;

$(AB)$ denotes the *distance* from $A$ to $B$, *sense* being relevant;

$\overline{AB}$ denotes the *vector* associated with the segment $AB$, in the sense from $A$ to $B$.

If $A$, $B$, $C$ are any three non-collinear points, then:

"The plane $ABC$," or simply $ABC$, means the *whole plane* containing $A$, $B$, $C$.

We use the abbreviations: w.r.t. ≡ "with respect to"; r.h.s. ≡ "right-hand side"; l.h.s. ≡ "left-hand side"; and a few others introduced subsequently.

## 2. Cartesian Coordinates

Consider any fixed point $O$ and any three distinct planes through $O$. These planes meet in pairs in three non-coplanar lines through $O$; let $X$, $Y$, $Z$ be fixed points, other than $O$, one on each line. Let $P$ be any point. The lines through $P$ parallel respectively to $OX$, $OY$, $OZ$ meet the planes $OYZ$, $OZX$, $OXY$ in points $L$, $M$, $N$, say (fig. 1).

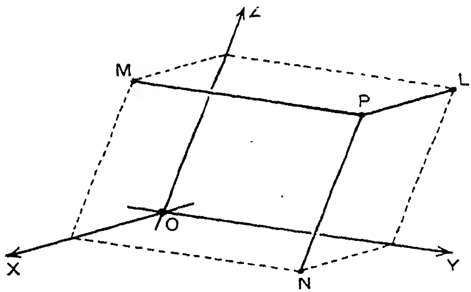


FIG. 1.

Write $x = |LP|$ if $P$, $X$ are on the same side of $OYZ$, $x = -|LP|$ if $P$, $X$ are on opposite sides of $OYZ$, lengths being measured in terms of some selected unit; let $y$, $z$ be analogously related to $|MP|$, $|NP|$. Then, when $P$ is given, the numbers $x$, $y$, $z$ are uniquely determined. Conversely, it is seen that, given any three positive or negative numbers $x$, $y$, $z$, there is a unique point $P$ with which these numbers can be associated in the manner described. So we may speak of $P$ as "the point $(x, y, z)$."

When the points of $\mathcal{E}$ are labelled in this fashion, we say that they are referred to *origin* $O$ and *coordinate planes* $OYZ$, $OZX$, $OXY$, or *coordinate axes* $OX$, $OY$, $OZ$. We call $x$, $y$, $z$ the (cartesian) *coordinates* of $P$ in this frame of reference.

Two features should be noted: (i) It must be realised that we can describe any point $P$ only by its relationship to some particular set of points *arbitrarily* chosen as a system of reference. (ii) When we say that $\mathcal{E}$ is *real* we mean merely that every point of $\mathcal{E}$ can be labelled with three *real numbers* serving as coordinates. Consequently we must ensure that any algebraic theorems to which we attempt to give geometrical interpretations do in fact hold good in the field of real numbers.

If $OX$, $OY$, $OZ$ are mutually perpendicular, we call them *rectangular* or *orthogonal* axes. $L$, $M$, $N$ are then the orthogonal projections of $P$ on the coordinate planes, and $x$, $y$, $z$ the perpendicular distances of $P$ from these planes, with appropriate signs attached. *Unless otherwise stated, we shall use only such rectangular cartesian coordinates.* Also, for definiteness, we shall use *right-handed* systems, *i.e.* viewing from $O$ towards $X$, a rotation from $Y$ towards $Z$ would be that of a right-handed screw, and so on in cyclic order.

1. $x$, $y$, $z$ are the rectangular components parallel to the axes of the vector $\overline{OP}$.

**Length of a segment.** *If* $P_1$, $P_2$ *are the points* $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, *then*

$$|P_1P_2|^2 = (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2. \quad . \quad (1)$$

This follows from an elementary application of Pythagoras's theorem to the rectangular parallelepiped having $P_1$, $P_2$ as opposite vertices and edges parallel to $OX$, $OY$, $OZ$.

2. If the axes are oblique and the angles $YOZ$, $ZOX$, $XOY$ are $\Theta$, $\Phi$, $\Psi$, then

$$|OP|^2 = x^2+y^2+z^2+2yz\cos\Theta+2zx\cos\Phi+2xy\cos\Psi.$$

3. A necessary and sufficient condition for a cartesian coordinate system to be rectangular is that $|OP|^2 = x^2+y^2+z^2$ for all positions of $P(x, y, z)$.

**Coordinates in general.** We shall use the term "coordinates" with the following general meaning: Let $\Sigma$ be a collection of geometrical objects such that every member of $\Sigma$ is labelled by a unique ordered set of $n$ numbers $(\xi, \eta, \ldots, \tau)$ and such that every such set of numbers, in a specified range, is the label of a unique member of $\Sigma$. Then $\xi, \eta, \ldots, \tau$ are called the *coordinates* of the corresponding member of $\Sigma$, in this system of labelling.

If the objects are points, we can when desirable distinguish their coordinates as "point-coordinates," if planes, as "plane-coordinates," and so on.

However, we sometimes find it convenient to replace the $n$ coordinates by the ratios of $n+1$ other numbers, or of more than $n+1$ numbers connected by certain specified relations. We shall also call these new numbers "coordinates," and shall find that we may do so without causing confusion.

## 3. Projections

Projection will be restricted to mean *orthogonal projection*. The projection of a point $P$ on a plane $\Pi$ is the foot of the normal from $P$ to $\Pi$. The projection of any other figure on $\Pi$ is the aggregate of the projections of its points, *e.g.* the projection of a line $s$ is the intersection of $\Pi$ with the plane through $s$ perpendicular to $\Pi$. The projection of a point $P$ on a line $s$ is the foot of the perpendicular from $P$ on $s$, *i.e.* the meet of $s$ with the plane through $P$ normal to $s$.

The angle $\theta$ between two planes $\Pi$, $\Lambda$ is the angle * between the normals from any point to $\Pi$, $\Lambda$. Let a closed boundary in $\Pi$ enclose area $\alpha$, and let its projection in $\Lambda$ enclose area $\beta$. Since lengths parallel to the intersection of $\Pi$, $\Lambda$ are unchanged by the projection, while lengths perpendicular to it are multiplied by $\cos\theta$, we find $\beta = \alpha\cos\theta$.

The angle $\phi$ between a line $s$ and a plane $\Pi$ is the angle *

* The acute angle, unless otherwise stated.

between $s$ and its projection on $\Pi$. Consequently the projection on $\Pi$ of a segment $PQ$ of $s$ has length $|PQ| \cos \phi$.

The angle $\psi$ between two skew lines $s$, $t$ is the angle * between lines through any point parallel to $s$, $t$. The projection on $t$ of a segment $PQ$ of $s$ has length $|PQ| \cos \psi$. But this should be carefully compared with the following paragraph.

**Sensed lines.** There are two opposed senses of displacement along a line $s$; we arbitrarily call one positive and the other negative. The positive one is sometimes called merely *the* sense of $s$. When we wish to emphasise that $s$ has an assigned sense we denote it by $\mathbf{s}$. $P_1, P_2$ being

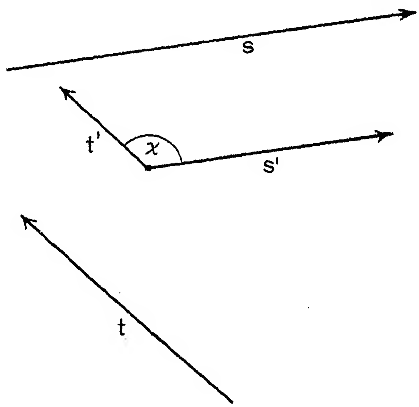


FIG. 2.

points of $\mathbf{s}$, we reckon the distance $(P_1P_2)$ positive if the displacement from $P_1$ to $P_2$ is in the positive sense, and otherwise negative. Thus, if $(P_1P_2)$ is positive, $(P_1P_2) = |P_1P_2| = -(P_2P_1)$. If $Q$ is another point of $\mathbf{s}$, we say that $Q$ divides the segment $P_1P_2$ in the ratio $(P_1Q) : (QP_2)$.

Let $\mathbf{t}$ be any other "sensed" line. We now define the

* The acute angle, unless otherwise stated.

angle $\chi$ between $\mathbf{s}$, $\mathbf{t}$ as the angle ($0 \leqslant \chi \leqslant \pi$) between lines $\mathbf{s}'$, $\mathbf{t}'$ through any point parallel to, *and in the same sense as*, $\mathbf{s}$, $\mathbf{t}$, respectively (fig. 2, where arrows indicate the senses).

1. The ratio in which $Q$ divides the segment $P_1$, $P_2$ is positive if $Q$ lies between $P_1$, $P_2$; negative and numerically less than unity if $P_1$ lies between $Q$, $P_2$; negative and numerically greater than unity if $P_2$ lies between $P_1$, $Q$.

2. $P$, $Q$ being points of $\mathbf{s}$, $P^*$, $Q^*$ their projections on $\mathbf{t}$, $(PQ)$ has a sign depending on the sense ascribed to $\mathbf{s}$, $(P^*Q^*)$ one depending on the sense ascribed to $\mathbf{t}$. We call $(P^*Q^*)$ the projection of $(PQ)$; in all cases $(P^*Q^*) = (PQ)\cos\chi$. [The problem is merely to see that, coupled with our definitions of $(PQ)$, $(P^*Q^*)$, $\chi$, this formula is equivalent to the standard definition of $\cos\chi$.]

3. $P_1P_2 \ldots P_nP_1$ being any polygon, not necessarily plane, the sum of the projections of its sides $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\ldots$ , $\overline{P_nP_1}$ on any (sensed) line is zero.

4. *The projection of a given vector on any (sensed) line is the sum of the projections of its components.*

## 4. Direction-cosines and Direction-ratios

Let $\mathbf{v}$ be any sensed line through $O$. We can conveniently describe its orientation by its relation to $S$, the sphere with centre $O$ and unit radius. For $\mathbf{v}$ meets $S$ at the ends of a diameter and one end, say $V$, is such that $(OV)$ is positive (fig. 3). If $\mathbf{v}$ is given, $V$ is a unique point of $S$; conversely, if $V$ is any given point of $S$, then $\mathbf{v}$ is uniquely determined as the line $\mathbf{OV}$. Let $V$ have coordinates $l, m, n$; it lies on $S$ if and only if $|OV| = 1$, *i.e.* from **1** (1),

$$l^2+m^2+n^2 = 1. \quad . \quad . \quad . \quad (1)$$

Therefore the preceding statement is equivalent to: *If* $\mathbf{v}$ *is given, then* $l, m, n$ *satisfying* (1) *are uniquely determined; if* $l, m, n$ *satisfying* (1) *are given, then* $\mathbf{v}$ *is uniquely determined.*

The numbers $l, m, n$ specify completely the direction, including the sense, of $\mathbf{v}$, and hence of *any line* $\mathbf{s}$ *parallel to* $\mathbf{v}$

*and in the same sense.* They are called the *direction-cosines* (d-c's) of **s**; (1) is the relation satisfied by every set of d-c's.

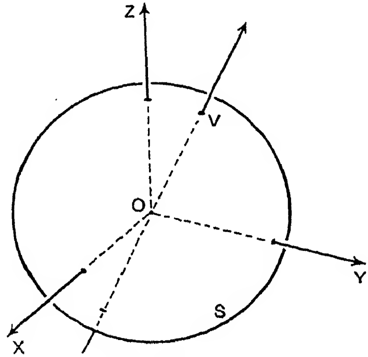


FIG. 3.

1. $l$, $m$, $n$ are the cosines of the angles $\alpha$, $\beta$, $\gamma$ (say) between **s** and **OX**, **OY**, **OZ**. $\alpha$, $\beta$, $\gamma$ are called the *direction-angles* of **s**.

2. $l$, $m$, $n$ are the components parallel to **OX**, **OY**, **OZ** of a unit vector along **s**.

3. If the sense of **s** is reversed, then the signs of its d-c's are reversed.

Now let $\lambda$, $\mu$, $\nu$ be a set of numbers proportional to $l$, $m$, $n$; then, using (1),

$$\frac{l}{\lambda} = \frac{m}{\mu} = \frac{n}{\nu} = \frac{\pm 1}{\sqrt{(\lambda^2+\mu^2+\nu^2)}},$$

*i.e.* (2)

$$(l, m, n) = \pm \frac{\lambda, \mu, \nu}{\sqrt{(\lambda^2+\mu^2+\nu^2)}}.$$

So, if $\lambda$, $\mu$, $\nu$ are given, $l$, $m$, $n$ are determined apart from sign, *i.e.* the direction of $s$ is determined apart from sense. These numbers are called *direction-ratios* (d-r's) of $s$.

4. $\lambda$, $\mu$, $\nu$ are the coordinates of some point of $v$. They are also the components of some vector along $s$.

5. D-r's of the join of $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ are $x_2-x_1$, $y_2-y_1$, $z_2-z_1$. The d-c's of $\mathbf{P}_1\mathbf{P}_2$ are these quantities divided by $|P_1P_2|$.

**Notation.** We write "the direction $(l, m, n)$" for the sensed direction having d-c's $l, m, n$; "the direction $(\lambda, \mu, \nu)$" for the unsensed direction having d-r's $\lambda, \mu, \nu$. When occasionally other symbols are used, and we write, for instance, "the direction $(a, b, c)$," then $a, b, c$ are to be interpreted as d-r's unless the context shows that d-c's are implied.

**Angle between two directions.** *The angle $\chi$ between two directions $(l, m, n)$, $(l', m', n')$ is given by*

$$\cos \chi = ll' + mm' + nn'. \quad (0 \leqslant \chi \leqslant \pi). \tag{3}$$

Let $V$, $V'$ be the *points* $(l, m, n)$, $(l', m', n')$; then $\overline{OV}$, $\overline{OV'}$ have the given directions. The projection of $\overline{OV}$ on $\overline{OV'}$ is $\cos \chi$, since $|OV| = 1$. The $x$-component of $\overline{OV}$ is $l$, and the cosine of the angle between $\overline{OX}$ and $\overline{OV'}$ is $l'$; hence the projection of the $x$-component of $\overline{OV}$ on $\overline{OV'}$ is $ll'$, and so on. So (3) results from **3** 4.

It follows that $(l, m, n)$, $(l', m', n')$ *are orthogonal if and only if*

$$ll' + mm' + nn' = 0. \tag{4}$$

6. (Another proof.) Using **1** (1), **4** (1), we have

$$|VV'|^2 = (l-l')^2 + (m-m')^2 + (n-n')^2 = 2 - 2(ll' + mm' + nn').$$

Again, from the triangle $OVV'$ in which $|OV| = |OV'| = 1$, $\angle VOV' = \chi$,

$$|VV'|^2 = |OV|^2 + |OV'|^2 - 2|OV||OV'|\cos \chi = 2 - 2\cos \chi.$$

Comparing these values, we recover (3).

7. $a, b, c, a', b', c'$ being any numbers,

$$(bc'-b'c)^2 + (ca'-c'a)^2 + (ab'-a'b)^2 \equiv (a^2+b^2+c^2)(a'^2+b'^2+c'^2) - (aa'+bb'+cc')^2.$$

8. From (3) and 7,

$$\sin \chi = +\sqrt{\{(mn'-m'n)^2 + (nl'-n'l)^2 + (lm'-l'm)^2\}}.$$

9. If $\psi$ is the angle between the directions $(\lambda, \mu, \nu)$, $(\lambda', \mu', \nu')$, then

$$\cos\psi = (\lambda\lambda'+\mu\mu'+\nu\nu') \div \sqrt{\{(\lambda^2+\mu^2+\nu^2)(\lambda'^2+\mu'^2+\nu'^2)\}};$$
$$\sin\psi = \sqrt{[\{(\mu\nu'-\mu'\nu)^2+(\nu\lambda'-\nu'\lambda)^2 + (\lambda\mu'-\lambda'\mu)^2\}/\{(\lambda^2+\mu^2+\nu^2)(\lambda'^2+\mu'^2+\nu'^2)\}]}.$$

[These being unsensed directions, the acute angle is taken; so the square roots are chosen so that $\cos\psi$, $\sin\psi$ are positive.]

10. The projection of the segment $P_1P_2$ on the direction $(\lambda, \mu, \nu)$ has length

$$|\,\lambda(x_2-x_1)+\mu(y_2-y_1)+\nu(z_2-z_1)\,| \,/\sqrt{(\lambda^2+\mu^2+\nu^2)}.\quad \text{[Use 5, 9.]}$$

11. The direction perpendicular to each of two distinct directions $(\lambda, \mu, \nu)$, $(\lambda', \mu', \nu')$ has d-r's $\mu\nu'-\mu'\nu$, $\nu\lambda'-\nu'\lambda$, $\lambda\mu'-\lambda'\mu$.

12. $\chi$ being the angle between $(l, m, n)$, $(l', m', n')$, the direction perpendicular to these has d-c's

$$\pm(mn'-m'n,\quad nl'-n'l,\quad lm'-l'm)\div\sin\chi.$$

The positive sign applies to the direction such that a right-handed screw travelling along it would turn from $(l, m, n)$ towards $(l', m', n')$.

13. The directions $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ are parallel to a single plane if and only if

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = 0.$$

## 5. Transformation of Coordinates

Our object is to establish properties of geometrical figures which are independent of any particular labelling of the points. Consequently we choose at each stage the labelling which best facilitates the calculations, and so we want to be able to change from one system of labelling to another. Restricting ourselves to rectangular cartesian coordinates, the general transformation is a combination of (i), (ii) as follows, provided the unit of length is unaltered.

(i) **Change of origin without rotation of axes.** Let $\mathcal{S}$, $\mathcal{S}^*$ be two coordinate systems, origins $O$, $O^*$, corre-

sponding axes being parallel and in the same sense. Let the coordinates of $O^*$ referred to $\mathcal{S}$ be $\xi$, $\eta$, $\zeta$. The coordinates of any point $P$ being $x, y, z$; $x^*, y^*, z^*$ referred to $\mathcal{S}$, $\mathcal{S}^*$ respectively, we have

$$x = x^*+\xi, \quad y = y^*+\eta, \quad z = z^*+\zeta. \quad . \qquad (1)$$

For these express merely the fact that the distance of $P$ from $OYZ$ is the sum of its distance from $O^*Y^*Z^*$ and that of $O^*Y^*Z^*$ from $OYZ$, and so on.

1. If $\rho$ is the position vector of $O^*$ referred to $O$, $\mathbf{r}$, $\mathbf{r}^*$ the position vectors of $P$ referred to $O$, $O^*$, then $\mathbf{r} = \mathbf{r}^*+\rho$, yielding (1).

(ii) **Rotation of axes without change of origin.** Let $OX, OY, OZ$ ($\mathcal{S}$), $OX', OY', OZ'$ ($\mathcal{S}'$) be two systems of axes, origin $O$. Let $OX'$, $OY'$, $OZ'$ have d-c's $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ referred to $\mathcal{S}$.

We have

$$l_1^2+m_1^2+n_1^2 = l_2^2+m_2^2+n_2^2 = l_3^2+m_3^2+n_3^2 = 1, \qquad (2)$$

and, the three directions being mutually perpendicular,

$$l_2l_3+m_2m_3+n_2n_3 = l_3l_1+m_3m_1+n_3n_1 = l_1l_2+m_1m_2+n_1n_2 = 0. \quad (3)$$

Also, since $l_1$, $l_2$, $l_3$ are the cosines of the angles between $OX$ and $OX'$, $OY'$, $OZ'$, they are the d-c's of $OX$ referred to $\mathcal{S}'$. Similarly $m_1$, $m_2$, $m_3$; $n_1$, $n_2$, $n_3$ are the d-c's of $OY$, $OZ$ referred to $\mathcal{S}'$. So, corresponding to (2), (3), we have

$$l_1^2+l_2^2+l_3^2 = m_1^2+m_2^2+m_3^2 = n_1^2+n_2^2+n_3^2 = 1, \qquad (4)$$

$$m_1n_1+m_2n_2+m_3n_3 = n_1l_1+n_2l_2+n_3l_3 = l_1m_1+l_2m_2+l_3m_3 = 0. \quad (5)$$

2. If

$$T = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}, \quad \text{then } T^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 1,$$

using (2), (3). Hence $T = \pm 1$.

3. If $\mathcal{S}$, $\mathcal{S}'$ coincide, then $T = +1$.

4. If $\mathcal{S}$, $\mathcal{S}'$ are both right-handed (r.h), or both left-handed

(l.h), then $T = +1$ for all $\mathcal{S}'$. Suppose $\mathcal{S}$ is r.h. (2)–(5) hold whether or not $\mathcal{S}'$ is also r.h. But if and only if $\mathcal{S}'$ is r.h could it be rotated into coincidence with $\mathcal{S}$. During the rotation, the value of the determinant $T$, if it changes at all, must do so continuously. But by 2 it can take only the values $\pm 1$; there being no possible intermediate values it cannot change continuously from one to the other. Hence it is *always* $+1$, or *always* $-1$. In the final position of coincidence with $\mathcal{S}$, $T = +1$, from 3. Therefore $T = +1$ for every position of $\mathcal{S}'$.

5. If $\mathcal{S}$, $\mathcal{S}'$ are one r.h and one l.h system, then $T = -1$.

6. $\mathcal{S}$, $\mathcal{S}'$ being both r.h or both l.h, we have

$$l_1 = m_2n_3 - m_3n_2, \text{ and so on.}$$

The coordinates of any point $P$ being $x, y, z$; $x', y', z'$ referred to $\mathcal{S}$, $\mathcal{S}'$, $x$ is the projection of $\overline{OP}$ on **OX**, $x', y', z'$ the components of $\overline{OP}$ parallel to **OX′**, **OY′**, **OZ′**, so that $l_1x', l_2y', l_3z'$ are the projections of these components on **OX**. Therefore, using **3** 4, and treating $y, z$ analogously, we have

$$\left.\begin{aligned} x &= l_1x' + l_2y' + l_3z', \\ y &= m_1x' + m_2y' + m_3z', \\ z &= n_1x' + n_2y' + n_3z', \end{aligned}\right\} \quad . \quad . \quad (6)$$

giving the required substitution. Symmetrically, or by solving (6) and using (2)–(5), we have

$$\left.\begin{aligned} x' &= l_1x + m_1y + n_1z, \\ y' &= l_2x + m_2y + n_2z, \\ z' &= l_3x + m_3y + n_3z, \end{aligned}\right\} \quad . \quad . \quad (7)$$

giving the inverse substitution.

7. Verify that (6), (7) both give $x^2+y^2+z^2 = x'^2+y'^2+z'^2$.

8. **Orthogonal matrices.** From **2** 3 it follows that the required transformation leaves $x^2+y^2+z^2$ invariant and so is "orthogonal." The above theory is then more concisely expressed in the language of orthogonal matrices, and the reader should compare Aitken, section **24**, particularly examples 2, 6, on this topic.

9. Any change of axes transforms a polynomial of degree $n$ in the original coordinates into one of degree $n$ in the new

coordinates. [For the substitutions (1), (6) cannot *raise* the degree; neither can they *lower* the degree, for, if they did, their inverses, which are similar transformations, would raise the degree.]

10. A change of axes cannot alter the number of polynomial factors of a given polynomial in $x$, $y$, $z$.

11. The general transformation of rectangular axes involves six *independent* parameters, or seven if the unit of length be changed.

CHAPTER II

# PLANES AND LINES

THIS chapter supplies an elementary account of analytic geometry of planes and lines; certain more general considerations await Chapter IV.

## 6. Collinear Points

*The point $P(x, y, z)$ which divides the join of $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ in the ratio $\lambda_1 : \lambda_2$ is given by*

$$x = \frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_2 + \lambda_1}, \quad y = \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_2 + \lambda_1}, \quad z = \frac{\lambda_2 z_1 + \lambda_1 z_2}{\lambda_2 + \lambda_1}. \tag{1}$$

This is proved in the same way as the corresponding theorem in two dimensions (or see 1 below).

It follows that $P$ lies on the join of $P_1$, $P_2$ if and only if numbers $\lambda_1$, $\lambda_2$ exist such that equations (1) are satisfied. Expressing this in a more symmetrical notation, *points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ are collinear if and only if numbers $\mu_1$, $\mu_2$, $\mu_3$, not all zero, exist such that*

$$\begin{aligned} \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 &= 0, \\ \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 &= 0, \\ \mu_1 z_1 + \mu_2 z_2 + \mu_3 z_3 &= 0, \\ \mu_1 + \mu_2 + \mu_3 &= 0. \end{aligned} \tag{2}$$

Note that if $P_1$, $P_2$, $P_3$ are collinear and distinct, then none of $\mu_1$, $\mu_2$, $\mu_3$ is zero.

Further, numbers $\mu_1$, $\mu_2$, $\mu_3$, not all zero, satisfying (2)

exist if and only if every 3-rowed determinant formed from the matrix

$$\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \quad . \quad . \quad . \qquad (3)$$

is zero (Aitken, **28**). This condition is commonly written

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \qquad (4)$$

giving the most concise form of the collinearity condition.

1. Let $\mathbf{P}_1\mathbf{P}_2$ have direction $(l, m, n)$; let $P_1^*$, $P_2^*$, $P^*$ be the projections of $P_1$, $P_2$, $P$ on $OX$. Then

$$x-x_1 = (P_1^*P^*) = (P_1P)l, \qquad x_2-x = (P^*P_2^*) = (PP_2)l,$$

and the first of equations (1) follows from

$$(P_1P) : (PP_2) = \lambda_1 : \lambda_2.$$

Similarly for the others. What happens if $l = 0$?

2. The present section (but not 1) is still valid if the axes are oblique.

3. If in (3) the elements of no column are proportional to those of another, then the vanishing of any *two* 3-rowed determinants formed from it ensures also the vanishing of the remaining two. In any case, if the three of these determinants each containing the last column are zero, then the determinant of the first three columns is also zero.

## 7. Coplanar Points

We prove: *Points* $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$ *are coplanar if and only if numbers* $\mu_1, \mu_2, \mu_3, \mu_4$, *not all zero, exist such that*

$$\left.\begin{aligned} \mu_1x_1+\mu_2x_2+\mu_3x_3+\mu_4x_4 &= 0, \\ \mu_1y_1+\mu_2y_2+\mu_3y_3+\mu_4y_4 &= 0, \\ \mu_1z_1+\mu_2z_2+\mu_3z_3+\mu_4z_4 &= 0, \\ \mu_1+\mu_2+\mu_3+\mu_4 &= 0. \end{aligned}\right\} \quad . \quad . \qquad (1)$$

Suppose $P_1$, $P_2$, $P_3$, $P_4$ are distinct; otherwise the theorem is trivial. The four points are coplanar if and only if the lines $P_1P_2$, $P_3P_4$ have a point $P(x, y, z)$, say, in common. If $P$ exists, then by **6** there exist numbers $\mu$, $\mu'$, $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$ such that

$$\left.\begin{aligned} \mu x+\mu_1x_1+\mu_2x_2 &= 0,\\ \mu y+\mu_1y_1+\mu_2y_2 &= 0,\\ \mu z+\mu_1z_1+\mu_2z_2 &= 0,\\ \mu+\mu_1+\mu_2 &= 0; \end{aligned}\right\} \quad (2) \qquad \left.\begin{aligned} \mu' x+\mu_3x_3+\mu_4x_4 &= 0,\\ \mu' y+\mu_3y_3+\mu_4y_4 &= 0,\\ \mu' z+\mu_3z_3+\mu_4z_4 &= 0,\\ \mu'+\mu_3+\mu_4 &= 0. \end{aligned}\right\} \quad (3)$$

Now $\mu \neq 0$; otherwise it would follow from (2) that $P_1 \equiv P_2$. Similarly $\mu' \neq 0$. Therefore, by absorbing a suitable constant into $\mu'$, $\mu_3$, $\mu_4$, we can without loss of generality suppose $\mu+\mu' = 0$. Then by adding corresponding equations in (2), (3) we obtain (1).

Conversely, assuming (1), we can reverse the algebra and derive equations of the form (2), (3), thus establishing the existence of $P$. Hence the theorem is proved.

Further, by a well-known theorem of algebra, numbers $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$, not all zero, satisfying (1) exist if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0, \qquad (4)$$

giving the most concise form of the coplanarity condition.

1. $P_4$ is the centroid of "masses" (positive or negative) $\mu_1$, $\mu_2$, $\mu_3$ at $P_1$, $P_2$, $P_3$.

## 8. General Equation of the First Degree

The general equation of the first degree (*linear* equation) in $x$, $y$, $z$ is

$$\Pi(x, y, z) \equiv ax+by+cz+d = 0, \qquad (1)$$

where $a$, $b$, $c$, $d$ are any given constants, $a$, $b$, $c$ being not all zero. The values of $x$, $y$, $z$ satisfying (1) depend only on

the ratios $a:b:c:d$, *i.e.* on *three independent constants.* There is an infinite number of points satisfying (1); call their aggregate $\Pi$. We say that $\Pi$ is the locus of equation (1), and that (1) is the equation of $\Pi$.

We here introduce the device of using a single symbol, in this case $\Pi$, for the locus of an equation and also for the l.h.s. of that equation, giving in this case the contracted form $\Pi = 0$. This proves useful, and is found to introduce no confusion.

1. Not all points of $\Pi$ are collinear.

*The equation of every plane is linear.* For let $\Lambda$ be a given plane and let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ be any three non-collinear points of $\Lambda$. Then $P(x, y, z)$ belongs to $\Lambda$ if and only if it is coplanar with $P_1$, $P_2$, $P_3$, *i.e.* from **7** (4), if and only if $x$, $y$, $z$ satisfy

$$\begin{array}{cccc} & y & z & \\ & y_1 & z_1 & \\ x_2 & & z_2 & \\ x_3 & y_3 & z_3 & \end{array} = 0. \tag{2}$$

This is therefore the equation of $\Lambda$ and is linear. The coefficients of $x, y, z$ are not all zero; otherwise, from **6** (3), $P_1$, $P_2$, $P_3$ would be collinear.

*The locus of every linear equation is a plane.* For, $\Pi$ being the locus of (1), let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ be three non-collinear points of $\Pi$ (*cf.* 1). Then solving the equations

$$\Pi(x_1, y_1, z_1) = 0, \quad \Pi(x_2, y_2, z_2) = 0, \quad \Pi(x_3, y_3, z_3) = 0,$$

for $a$, $b$, $c$, $d$ we get

$$a:b:c:d = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} : -\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} : \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} : -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{3}$$

From **6**, these determinants are not all zero since $P_1$, $P_2$, $P_3$ are not collinear. Hence, substituting these values for

$a$, $b$, $c$, $d$ in (1), it assumes the form (2). So $\Pi$ is the plane $P_1P_2P_3$.

**Rectified equation.** Let $\Pi$ be a plane, $P(x, y, z)$ any point of $\Pi$, $N$ the foot of the normal from $O$ to $\Pi$. Also let $(l, m, n)$ be the direction of $\overline{ON}$ or $\overline{NO}$, and let $(ON) = p$ (this being positive or negative according as $(l, m, n)$ is the direction of $\overline{ON}$ or $\overline{NO}$). Expressing the fact that $(ON)$ is the projection of $(OP)$ on **ON**, we have

$$lx+my+nz = p. \tag{4}$$

This is the equation of $\Pi$ in what is called *rectified* form.

Conversely, if $\Pi$ is given by (1), then comparison with (4) gives

$$l/a = m/b = n/c = -p/d,$$

whence

$$(ON) = -d/\sqrt{(a^2+b^2+c^2)},$$

measured in the direction (5)

$$(a, b, c)\div\sqrt{(a^2+b^2+c^2)}.$$

Therefore $a$, $b$, $c$ *are direction-ratios of any normal to* $\Pi$; if (1) is written so that $d$ is negative, then

$$(a, b, c)\div\sqrt{(a^2+b^2+c^2)}$$

..c the direction-cosines of **ON**.

2. $M$ being the foot of the normal from any point $Q(x', y', z')$ to the plane (4), we have $(MQ) = lx'+my'+nz'-p$, measured in the direction $(l, m, n)$. [For $(MQ)$ = projection of $(OQ)$ on $(l, m, n)$—projection of $(OM)$ on $(l, m, n)$.]

3. Using (5) and 2, *the perpendicular distance of $Q$ from the plane* (1) *is*

$$(ax'+by'+cz'+d)\div\sqrt{(a^2+b^2+c^2)},$$

this expression being positive for all $Q$ in the region on one side of $\Pi$, negative for all $Q$ in the region on the other side.

4. Form the equations of the planes bisecting the angles between two given planes.

5. The plane making intercepts $\alpha$, $\beta$, $\gamma$ on $OX$, $OY$, $OZ$ has equation

$$x/\alpha+y/\beta+z/\gamma = 1.$$

6. The plane through $(x', y', z')$ parallel to two given directions $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ is

$$\begin{array}{cc} x-x' & y-y \\ \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{array} \qquad = 0.$$

7. **Area of triangle.** Let $\alpha$ be the area of the triangle with vertices $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$; let the direction $(l, m, n)$ be normal to $BCD$. Then $(\pm)\alpha l$ is the area of the projection of $\triangle BCD$ on $OYZ$. But the projections of $B$, $C$, $D$ on $OYZ$ are the points $(y_2, z_2)$, $(y_3, z_3)$, $(y_4, z_4)$ in $OYZ$; so, by a theorem of plane geometry, the area of the projection is

$$(\pm)\tfrac{1}{2} \begin{array}{ccc} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{array}$$

Using corresponding results for $OZX$, $OXY$, we have, since $l^2+m^2+n^2 = 1$,

$$4\alpha^2 = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \tag{6}$$

8. **Volume of tetrahedron.** Let $\mathcal{V}$ be the volume of the tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$. The equation of $BCD$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \equiv x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0. \tag{7}$$

From 3, the perpendicular distance $p_1$ of $A$ from $BCD$ is got by writing $x_1$, $y_1$, $z_1$ for $x$, $y$, $z$ in l.h.s. of (7) and dividing by the square root of the sum of the squares of the coefficients of

$x$, $y$, $z$. By (6), the latter is $2a$. Hence, since $\mathcal{V} = \frac{1}{3}p_1 a$, we obtain

$$\mathcal{V} = \tfrac{1}{6} \begin{array}{cccc} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{array} \qquad (8)$$

This has been derived without regard to sign. An argument like that giving the sign of $T$ in **5** 4 shows: The axes being r.h, if a rotation in the sense $B \to C \to D$ carries a r.h screw towards $A$, then $\mathcal{V}$ given by (8) is positive, in the contrary case negative.

9. Recover (8) by methods analogous to those used in plane geometry to get the corresponding result for the area of a triangle.

10. The volume of the tetrahedron whose faces have equations $a_1x+b_1y+c_1z+d_1 = 0$, $a_2x+\ldots = 0$, $a_3x+\ldots = 0$, $a_4x+\ldots = 0$ is $K^3/6D_1D_2D_3D_4$, where $K$ is the determinant $|\, a_1\, b_2\, c_3\, d_4 \,|$ and $D_1, \ldots$ the cofactors of $d_1, \ldots$ in $K$. [We use the familiar device of indicating a determinant by its principal diagonal. Express the coordinates of the vertices in terms of cofactors of $K$, substitute in (8), and apply Jacobi's theorem (Aitken, **42**).]

11. In the tetrahedron $ABCD$, let $a$, $b$, $c$ be the sides of $\triangle ABC$, $d$, $e$, $f$ the edges joining $A$, $B$, $C$ to $D$. Take $D$ as origin, then (8) gives

$$288\mathcal{V}^2 = 8 \begin{array}{ccc} x_1 & y_1 & \\ x_2 & & z_2 \\ x_3 & y_3 & z_3 \end{array} \quad \begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \\ & z_2 \end{array}$$

$$\begin{array}{ccc} x_1^2+y_1^2+z_1^2 & x_1x_2+y_1y_2+z_1z_2 & x_1x_3+y_1y_3+z_1z_3 \\ x_2x_1+y_2y_1+z_2z_1 & x_2^2+y_2^2+z_2^2 & x_2x_3+y_2y_3+z_2z_3 \\ x_3x_1+y_3y_1+z_3z_1 & x_3x_2+y_3y_2+z_3z_2 & x_3^2+y_3^2+z_3^2 \end{array}$$

$$\left| \begin{array}{ccc} 2d^2 & d^2+e^2-c^2 & d^2+f^2-b^2 \\ d^2+e^2-c^2 & 2e^2 & e^2+f^2-a^2 \\ d^2+f^2-b^2 & e^2+f^2-a^2 & 2f^2 \end{array} \right|$$

thus expressing the volume of a tetrahedron in terms of its edges. [This example is given as an application of the theory to establish a result independent of any coordinate system.]

Hence, or otherwise, express $\mathcal{V}$ in terms of three concurrent edges and the face-angles at the common vertex.

## 9. Incidence of Planes

Let four given distinct planes be

$$\Pi_1 \equiv a_1x+b_1y+c_1z+d_1 = 0, \quad . \quad . \quad (1)$$
$$\Pi_2 \equiv a_2x+b_2y+c_2z+d_2 = 0, \quad . \quad . \quad (2)$$
$$\Pi_3 \equiv a_3x+b_3y+c_3z+d_3 = 0, \quad . \quad . \quad (3)$$
$$\Pi_4 \equiv a_4x+b_4y+c_4z+d_4 = 0. \quad . \quad . \quad (4)$$

Suppose $\Pi_1$, $\Pi_2$ are not parallel; then they have in common a line $s_{12}$, say, and this must consist of all points satisfying (1), (2) simultaneously. Now consider the equation

$$\Pi \equiv k_1\Pi_1+k_2\Pi_2 = 0, \quad . \quad . \quad (5)$$

where $k_1$, $k_2$ are any constants. (5) is linear, so $\Pi$ is a plane. Also $\Pi = 0$ for all $(x, y, z)$ for which $\Pi_1 = 0$, $\Pi_2 = 0$ simultaneously; therefore $\Pi$ contains $s_{12}$. Further, $Q(x', y', z')$ being any point not on $s_{12}$, $\Pi$ contains $Q$ if

$$k_1(a_1x'+b_1y'+c_1z'+d_1)+k_2(a_2x'+b_2y'+c_2z'+d_2) = 0,$$

and this determines $k_1 : k_2$ uniquely. Since any plane through $s_{12}$ is determined when one of its points not on $s_{12}$ is given, it follows that *every plane through the intersection of* $\Pi_1$, $\Pi_2$ *is expressible in the form* (5).

1. $\Pi_1$, $\Pi_2$, $\Pi_3$ contain a common line, or are parallel, if and only if non-zero numbers $k_1$, $k_2$, $k_3$ exist such that

$$k_1\Pi_1+k_2\Pi_2+k_3\Pi_3 \equiv 0.$$

2. $\Pi_1$, $\Pi_2$, $\Pi_3$ are parallel to some line if and only if $| a_1 \; b_2 \; c_3 | = 0$.

3. The planes $\nu y-\mu z = p$, $\lambda z-\nu x = q$, $\mu x-\lambda y = r$ contain a common line if and only if $\lambda p+\mu q+\nu r = 0$. This line lies also in the plane $px+qy+rz = 0$.

Suppose $\Pi_3$ is not parallel to $s_{12}$. Then it meets $s_{12}$ in a point $P_4$, which is the only point common to $\Pi_1$, $\Pi_2$, $\Pi_3$; its coordinates are got by solving (1), (2), (3).

4. Every plane $\Pi$ through $P_4$ can be written

$$\Pi \equiv k_1\Pi_1+k_2\Pi_2+k_3\Pi_3 = 0,$$

where $k_1$, $k_2$, $k_3$ are constants. (Contrast 1.)

Finally, $\Pi_4$ also contains $P_4$, *i.e.* the four planes have at least one common point, if and only if (Aitken, **30**)

$$| a_1 \quad b_2 \quad c_3 \quad d_4 | = 0. \qquad . \qquad . \qquad (6)$$

5. If $\Pi_1$, $\Pi_2$, $\Pi_3$, $\Pi_4$ are not concurrent, and no three are parallel, then *any* plane $\Pi$ can be written

$$\Pi \equiv k_1\Pi_1+k_2\Pi_2+k_3\Pi_3+k_4\Pi_4 = 0,$$

where $k_1$, ... are constants.

6. **Quadriplanar coordinates.** $\Pi_1$, $\Pi_2$, $\Pi_3$, $\Pi_4$ being as in 5, constants $\alpha_1$, $\alpha_2$, ..., $\delta_4$ exist such that

$$x \equiv \alpha_1\Pi_1+\alpha_2\Pi_2+\alpha_3\Pi_3+\alpha_4\Pi_4, \qquad . \qquad . \qquad (7.1)$$
$$y \equiv \beta_1\Pi_1+\beta_2\Pi_2+\beta_3\Pi_3+\beta_4\Pi_4, \qquad . \qquad . \qquad (7.2)$$
$$z \equiv \gamma_1\Pi_1+\gamma_2\Pi_2+\gamma_3\Pi_3+\gamma_4\Pi_4, \qquad . \qquad . \qquad (7.3)$$
$$1 \equiv \delta_1\Pi_1+\delta_2\Pi_2+\delta_3\Pi_3+\delta_4\Pi_4. \qquad . \qquad . \qquad (7.4)$$

In fact, the condition that (6) is not satisfied makes possible such a unique solution of $a_1x+b_1y+c_1z+d_11 \equiv \Pi_1$, etc. (7.1)–(7.3) provide a transformation from $x$, $y$, $z$ to new "coordinates" $\Pi_1$, $\Pi_2$, $\Pi_3$, $\Pi_4$ connected by (7.4). Any polynomial of degree $n$ in $x, y, z$ is expressible as a *homogeneous* polynomial of degree $n$ in $\Pi_1$, $\Pi_2$, $\Pi_3$, $\Pi_4$.

Now suppose (1)–(4) to be in rectified form, so that $\Pi_1$, ... are the perpendicular distances of $P(x, y, z)$ from these planes; let $\Pi_4$ be positive when $P$, $P_4$ are on the same side of $\Pi_4$, and so on. $\Pi_1$, ... are then called *quadriplanar coordinates* of $P$, analogous to trilinear coordinates in two dimensions. The four planes enclose a tetrahedron; $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$ being the areas of its faces and $\mathcal{V}$ its volume, we have by elementary geometry $(\Pi_1\Delta_1+\Pi_2\Delta_2+\Pi_3\Delta_3+\Pi_4\Delta_4)/3\mathcal{V} = 1$, showing the geometrical meaning of (7.4).

## 10. Equations of a Line

We have seen that all the points which satisfy simultaneously two linear equations form a line. Conversely,

let $s$ be a given line and $\Pi_1$, $\Pi_2$ any two distinct planes through $s$. Then a point lies on $s$ if and only if it satisfies simultaneously

$$\Pi_1 = 0, \qquad \Pi_2 = 0. \qquad . \qquad . \qquad (1)$$

So we call these equations, taken together, the *equations of* $s$. These equations are not unique, but may be replaced by those of any other pair of planes through $s$. Some standard forms are given in 2–4 below.

This is our first instance of a locus being specified by more than one equation. But a single algebraic equation cannot suffice to specify a line. For there are in general infinitely many values of $x$, $y$ (say) satisfying such an equation for any given value of $z$, whereas on a line there is in general a unique point for which the coordinate $z$ has a given value. Therefore at least two equations are required. So we could not specify a line more simply than by two *linear* equations, and we have seen that these suffice.

In the field of real numbers, a single equation

$$f_1^2 + f_2^2 + \qquad + f_n^2 = 0$$

is, in fact, equivalent to the set $f_1 = 0$, $f_2 = 0$, ..., $f_n = 0$. But when we speak of a single equation we shall exclude such cases.*

1. The equations of a line depend on *four* independent constants.

2. $s$ being not parallel to $z = 0$, its equations can be put in the form

$$x = \alpha z + \alpha', \qquad y = \beta z + \beta'.$$

These define $s$ by its projections in the planes $y = 0$, $x = 0$.

3. The line joining two given points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

* Except where they arise as certain special cases of the equation of the second degree.

4. The line through the point $(x', y', z')$ in the direction $(\lambda, \mu, \nu)$ is, using 4 5,

$$\frac{\quad}{\lambda} = \frac{y-y'}{\mu} = \frac{z-z'}{\nu}$$

5. The line in 4 lies in the plane $ax+by+cz+d = 0$ if

$$a\lambda+b\mu+c\nu = 0, \qquad ax'+by'+cz'+d = 0.$$

[Algebraically: put each ratio in 4 equal to $t$ and make $(x, y, z)$ lie in the plane for all $t$. Geometrically: the direction $(\lambda, \mu, \nu)$ must be parallel to the plane, and the point $(x', y', z')$ must be in the plane.]

6. State conditions for the line in 2 to lie in a given plane.

7. Show how to write the intersection of

$$a_1x+b_1y+c_1z+d = 0, \qquad a_2x+b_2y+c_2z+d = 0$$

in the form given in 4. [$(\lambda, \mu, \nu)$ being perpendicular to $(a_1, b_1, c_1)$, $(a_2, b_2, c_2)$ is given by

$$(b_1c_2-b_2c_1, \quad c_1a_2-c_2a_1, \quad a_1b_2-a_2b_1).$$

If then, say, $\nu \neq 0$, give $z$ any value $z'$ and solve for $x$, $y$, thus obtaining a point $(x', y', z')$ on the line.]

8. The perpendicular distance of $Q(x'', y'', z'')$ from the line through $P(x', y', z')$ with direction $(l, m, n)$ is

$$\sqrt{\{\Sigma[(y'-y'')n-(z'-z'')m]^2\}}.$$

[This is $|PQ| \sin \psi$ where $\psi$ is the angle between $PQ$ and the line.]

**Shortest distance between two lines.** Let two non-parallel lines $s_1$, $s_2$ be given by

$$(x-x_1)/\lambda_1 = (y-y_1)/\mu_1 = (z-z_1)/\nu_1, \qquad (1)$$

$$(x-x_2)/\lambda_2 = (y-y_2)/\mu_2 = (z-z_2)/\nu_2. \qquad (2)$$

The single direction $\delta$ perpendicular to $s_1$, $s_2$ is $(\mu_1\nu_2-\mu_2\nu_1, \nu_1\lambda_2-\nu_2\lambda_1, \lambda_1\mu_2-\lambda_2\mu_1)$. The plane $\Pi_1$ containing $s_1$ and

parallel to $\delta$, and the plane $\Pi_2$ containing $s_2$ and parallel to $\delta$ are, respectively, (8 6)

$$\left.\begin{aligned}\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \lambda_1 & \mu_1 & \nu_1 \\ \mu_1\nu_2-\mu_2\nu_1 & \nu_1\lambda_2-\nu_2\lambda_1 & \lambda_1\mu_2-\lambda_2\mu_1 \end{vmatrix} = 0, \\ \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ \lambda_2 & \mu_2 & \nu_2 \\ \mu_1\nu_2-\mu_2\nu_1 & \nu_1\lambda_2-\nu_2\lambda_1 & \lambda_1\mu_2-\lambda_2\mu_1 \end{vmatrix} = 0.\end{aligned}\right\} \qquad (3)$$

Since $\Pi_1$, $\Pi_2$ are both parallel to $\delta$, their line of intersection $s$ is parallel to $\delta$; since $\Pi_1$ contains $s_1$, $s$ meets $s_1$; similarly $s$ meets $s_2$. Therefore (3) are the equations of *a unique line $s$ perpendicular to $s_1$, $s_2$ and meeting them both* (in $N_1$, $N_2$, say).

Let $P_1$, $P_2$ be any points of $s_1$, $s_2$ respectively; let $\psi$ be the angle between $P_1P_2$, $N_1N_2$. Then $\psi = 0$ only if $P_1 \equiv N_1$, $P_2 \equiv N_2$; otherwise $P_1P_2$ would be a second line perpendicular to $s_1$, $s_2$ and meeting both. Now $P_1N_1$, $P_2N_2$ being perpendicular to $N_1N_2$, the segment $N_1N_2$ is the projection of the segment $P_1P_2$ on $N_1N_2$. So $|N_1N_2| = |P_1P_2| \cos\psi$, whence $|N_1N_2| < |P_1P_2|$ unless $P_1 \equiv N_1$, $P_2 \equiv N_2$. Therefore $|N_1N_2|$ is the *shortest distance* between $s_1$, $s_2$, as is well known from elementary geometry.

Taking in particular $P_1$, $P_2$ to be $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and using 4 10, the projection of the segment $P_1P_2$ on $s$ is

$$(\pm)\{\Sigma(x_2-x_1)(\mu_1\nu_2-\mu_2\nu_1)\}.\{\Sigma(\mu_1\nu_2-\mu_2\nu_1)^2\}^{-1/2},$$

*i.e.*

$$(\pm)\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} \div \sqrt{\{(\mu_1\nu_2-\mu_2\nu_1)^2+(\nu_1\lambda_2-\nu_2\lambda_1)^2 + (\lambda_1\mu_2-\lambda_2\mu_1)^2\}}, \qquad (4)$$

giving the length of the shortest distance. It follows that *$s_1$, $s_2$ intersect if and only if*

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0. \qquad (5)$$

9. Recover (4) by forming the equation of the plane through $s_1$ parallel to $s_2$ and finding the perpendicular distance of $(x_2, y_2, z_2)$ from it.

10. If $P_1$, $P_2$ move on two fixed lines so that $|P_1P_2|$ is constant, then $P_1P_2$ makes a constant angle with a fixed direction. [For $\psi$ defined above is constant.]

The equations of $s_1$, $s_2$ can now be put in a simple form which facilitates the solution of particular problems. Take origin $O$ the midpoint of $N_1N_2$; $OZ$ along $N_1N_2$; $OX$, $OY$ parallel to the bisectors of the angles between $s_1$, $s_2$. Then the equations of $s_1$, $s_2$ become

$$y = mx, \quad z = k; \qquad y = -mx, \quad z = -k, \tag{6}$$

where $2k = |N_1N_2|$, and $2\tan^{-1} m$ is the angle between $s_1$, $s_2$.

11. Find the locus of a variable line $s$ meeting fixed lines $s_1$, $s_2$ in $P_1$, $P_2$ so that $|N_1P_1| = |N_2P_2|$, $N_1$, $N_2$ being as defined above. [Taking axes as in (6), $P_1$, $P_2$ have coordinates of the forms $(x_1, mx_1, k)$, $(x_2, -mx_2, -k)$. The condition $|N_1P_1| = |N_2P_2|$ gives $x_1 = \pm x_2$. With $x_1 = x_2$, $P_1P_2$ is $(x-x_1)/0 = (y-mx_1)/mx_1 = (z-k)/k$. Eliminating $x_1$, the required locus is $mzx-ky = 0$, referred to this particular coordinate system. With $x_1 = -x_2$, it is $yz-kmx = 0$.

Later work will show that each locus is a hyperbolic paraboloid, and that this could be foreseen geometrically.]

## 11. Parametric Representation

The reader will recall the introduction of parametric (or "freedom") equations of a locus in two dimensions and the advantages of their use. Familiar instances are the representation of a line by equations of the form $x = a+pt$, $y = b+qt$, and of a parabola by $x = at^2$, $y = 2at$. Each of these is an example of a rational representation, having the fundamental property that it establishes a one-to-one correspondence between the points of the locus and the values of the parameter $t$. So we may specify completely

a point of the locus by the corresponding value of $t$, referring to it, if we wish, as "the point $t$." One and only one parameter is needed because a point on the locus has just one degree of freedom.

In three dimensions a *curve* (a line being the simplest case) is still by definition a locus on which a variable point has a single degree of freedom. The coordinates of such a point must again be expressible in terms of *one* parameter. A *surface* (a plane being the simplest case) is by definition a locus on which a variable point has two degrees of freedom. The coordinates of such a point must be expressible in terms of *two* independent parameters. In each case the expression of the coordinates in terms of the parameter or parameters need not be rational, or even algebraic. But if it can be made rational, then the curve or surface is said to be rational. Obviously, not every parametric representation of a rational locus is rational. Various features of these conceptions will receive subsequent illustration.

1. An arbitrary point in $\mathcal{E}$ has three degrees of freedom; a point satisfying a single equation has two degrees of freedom; a point satisfying two equations has in general one degree of freedom; a point satisfying three equations has in general no freedom, but must be one of a finite number (if the equations are algebraic) of fixed points. [Interpreted geometrically these results are: the locus in $\mathcal{E}$ of a single equation is a surface; if two surfaces intersect they do so, in general, in a curve; if a curve and a surface intersect they do so, in general, in isolated points. There are, however, important cases of exception, *e.g.* two or more surfaces may have a portion of each *surface* in common, three or more surfaces may have a *curve* in common. But the analytical formulation of these cases is too difficult to be given here.]

**Line.** Let $s$ be a given line; $l, m, n$ its d-c's; $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ fixed points of $s$; $P(x, y, z)$ any point of $s$.

Let $(P_1P) = r$, measured in the sense $(l, m, n)$. Then there is a one-to-one correspondence between the positions

of $P$ and the values of $r$ $(-\infty < r < \infty)$. Projecting $(P_1P)$ on the axes, we find

$$x = x_1 + lr, \quad y = y_1 + mr, \quad z = z_1 + nr. \quad . \qquad (1)$$

Again, let $(P_1P) : (PP_2) = \rho$. There is a one-to-one correspondence between the positions of $P$ (except $P \equiv P_2$) and the values of $\rho$ $(-\infty < \rho < \infty;\ \rho \neq -1)$. We have from **6**

$$\left.\begin{aligned} x = (x_1 + \rho x_2)/(1+\rho), \qquad y = (y_1 + \rho y_2)/(1+\rho),\\ z = (z_1 + \rho z_2)/(1+\rho). \end{aligned}\right\} \qquad (2)$$

(1), (2) give two *parametric representations of* $s$ in terms of $r$, $\rho$ respectively. They are particular cases of equations of the form

$$x = \frac{at+a'}{kt+k'}, \quad y = \frac{bt+b'}{kt+k'}, \quad z = \frac{ct+c'}{kt+k'}, \quad . \qquad (3)$$

where $a, a', \ldots, k'$ are constants, and $t$ is a variable parameter. (3) can be proved equivalent to the most general algebraic representation of a line $s$. But further discussion is best given in connexion with homographic correspondence, which is beyond our present scope. It should be noted,* however, that no value of $t$ corresponds to the point $(a/k, b/k, c/k)$, which is a point of $s$ if $k \neq 0$. To get a correspondence applying to *every* point of $s$ we must have $k = 0$, $k' \neq 0$.

2. There is a (1–1) correspondence between the values of $r$, $\rho$ in (1), (2). $[r = r_2\rho/(1+\rho)$, where $(P_1P_2) = r_2$.$]$

3. If lines $s$, $s^*$ are represented by (3), (1), and if we take $t \equiv r$, we establish a (1–1) correspondence between the points of $s$, $s^*$, *i.e.* we "map" $s$ on $s^*$.

4. If in (3) we put $t = (\alpha t^* + \beta)/(\gamma t^* + \delta)$, $(\alpha\delta \neq \beta\gamma)$, we obtain a representation of the same form in terms of $t^*$.

* This difficulty is easily overcome by using *homogeneous parameters* $t$, $t'$ and writing (3) as $x = (at + a't')/(kt + k't')$, etc.; then we use the ratio $t : t'$ in place of the $t$ in the text. This is equivalent to what is done in Chapter IV, but here we prefer a single symbol to emphasise the single degree of freedom. Analogous remarks apply to the case of the plane.

**Plane.** Let $\Pi$ be a given plane $(l, m, n)$, $(l', m', n')$ two distinct directions in $\Pi$; $P_1(x_1, y_1, z_1)$ $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ fixed non-collinear points of $\Pi$; $P(x, y, z)$ any point of $\Pi$.

Let $P$ have cartesian coordinates $r$, $r'$ referred to axes in $\Pi$ itself with origin $P_1$ and directions $(l, m, n)$, $(l', m', n')$, not necessarily orthogonal. Then there is a one-to-one correspondence between the positions of $P$ and the *pairs* of numbers $(r, r')$, $(-\infty < r, r' < \infty)$. Projecting $(P_1P)$ on $OX$, $OY$, $OZ$, we find

$$x = x_1 + lr + l'r', \quad y = y_1 + mr + m'r', \quad z = z_1 + nr + n'r'. \quad (4)$$

Again, let $P$ be the centroid of masses $\mu_1$, $\mu_2$, $\mu_3$ at $P_1$, $P_2$, $P_3$, and put $\mu_2/\mu_1 = \rho$, $\mu_3/\mu_1 = \sigma$. There is a one-to-one correspondence between the positions of $P$ (except on $P_2P_3$) and the *pairs* of numbers $(\rho, \sigma)$, $(-\infty < \rho, \sigma < \infty;$ $\rho + \sigma \neq -1)$. We have from **7**

$$\left.\begin{aligned} x &= (x_1 + \rho x_2 + \sigma x_3)/(1 + \rho + \sigma), \\ y &= (y_1 + \rho y_2 + \sigma y_3)/(1 + \rho + \sigma), \\ z &= (z_1 + \rho z_2 + \sigma z_3)/(1 + \rho + \sigma). \end{aligned}\right\} \quad . \quad (5)$$

(4), (5) give two *parametric representations of* $\Pi$ in terms of $(r, r')$, $(\rho, \sigma)$ respectively. They are particular cases of equations of the form

$$x = \frac{au + a'v + a''}{ku + k'v + k''}, \quad y = \frac{bu + b'v + b''}{ku + k'v + k''}, \quad z = \frac{cu + c'v + c''}{ku + k'v + k''}, \quad (6)$$

where $a, a', \ldots, k''$ are constants, and $u$, $v$ variable parameters. (6) can be proved equivalent to the most general algebraic parametric representation of a plane $\Pi$. But it should be noted that no values of $u$, $v$ correspond to any point on the join of $(a/k, b/k, c/k)$, $(a'/k', b'/k', c'/k')$, which are points of $\Pi$ if $k, k' \neq 0$. To get a correspondence applying to *every* point of $\Pi$ we must have $k = k' = 0$, $k'' \neq 0$.

5. There is a (1–1) correspondence between the values of $(r, r')$, $(\rho, \sigma)$ in (4), (5). [$r = (\rho r_2 + \sigma r_3)/(1 + \rho + \sigma)$, $r' = (\rho r_2' + \sigma r_3')/(1 + \rho + \sigma)$, where $P_2$, $P_3$ are $(r_2, r_2')$, $(r_3, r_3')$ in (4).]

6. If planes $\Pi$, $\Pi^*$ are represented by (6), (4), and if we take $(u, v) \equiv (r, r')$, we establish a (1–1) correspondence between the points of $\Pi$, $\Pi^*$, *i.e.* we "map" $\Pi$ on $\Pi^*$.

## 12. Plane-coordinates

We found that to any given plane there corresponds an equation $ax+by+cz+d = 0$ for which the ratios $a : b : c : d$ are uniquely determined, and to any such equation there corresponds a unique plane. Accordingly (from **2**) these ratios serve as "coordinates" for labelling planes in $\mathcal{E}$. Actually, however, we prefer to call $a$, $b$, $c$, $d$ themselves the *coordinates* of the corresponding plane, while recognising that the coordinates $ka$, $kb$, $kc$, $kd$ represent the same plane for all values of $k$ ($\neq 0$). Note that in $\mathcal{E}$ there is no plane with $a = b = c = 0$.

1. All the planes whose coordinates satisfy a given homogeneous linear equation $\alpha a+\beta b+\gamma c+\delta d = 0$ contain a fixed point [the point $(\alpha/\delta, \beta/\delta, \gamma/\delta)$], and conversely. This is the *equation of the point* in plane-coordinates.

2. All the planes whose coordinates satisfy two given homogeneous linear equations contain a fixed line [the join of the two points whose equations are given]. The two equations are called the *equations of the line* in plane-coordinates.

3. Give the theory analogous to **9**, replacing point- by plane-coordinates.

## 13. Line-coordinates

The case of a plane is straightforward because its equation depends in a simple way on three independent constants which suffice to determine it. But, if we want to specify a line $s$ in some corresponding manner, we observe that its equations in the symmetrical form

$$(x-x')/\lambda = (y-y')/\mu = (z-z')/\nu \qquad . \qquad (1)$$

contain *six* constants which do not depend quite simply on a set of *four* independent constants sufficient to determine $s$.

In particular, $(x', y', z')$ may be *any* fixed point on $s$. However, we obtain from (1)

$$\left.\begin{array}{c}\nu y-\mu z=\nu y'-\mu z', \quad \lambda z-\nu x=\lambda z'-\nu x', \\ \mu x-\lambda y=\mu x'-\lambda y',\end{array}\right\} . \qquad (2)$$

so that, $(x'', y'', z'')$ being any other point of $s$, we have $\nu y''-\mu z''=\nu y'-\mu z'$, etc. Hence, writing

$$\nu y'-\mu z'=p, \quad \lambda z'-\nu x'=q, \quad \mu x'-\lambda y'=r, \quad . \qquad (3)$$

the values of $p$, $q$, $r$ are the same for all positions of $(x', y', z')$ on $s$. Also from (3) we have identically

$$\lambda p+\mu q+\nu r=0. \quad . \quad . \quad . \qquad (4)$$

Finally, $s$ is unchanged if we multiply $\lambda$, $\mu$, $\nu$, and consequently $p$, $q$, $r$, by any non-zero constant. Therefore, to a given line there corresponds a set of six numbers $\lambda$, $\mu$, $\nu$, $p$, $q$, $r$, whose ratios are uniquely determined and which satisfy the identity (4), thus involving *four* independent quantities. Conversely, given any set of values of $\lambda, \mu, \nu, p, q, r$ satisfying (4), we can form the equations (2), *i.e.*

$$\nu y-\mu z=p, \quad \lambda z-\nu x=q, \quad \mu x-\lambda y=r, \quad . \qquad (5)$$

and these determine a unique line (**9** 3). We call $\lambda$, $\mu$, $\nu$, $p$, $q$, $r$ the *coordinates* of the line.*

1. Express the line-coordinates of the join of two given points in terms of the coordinates of these points.

2. Express the line-coordinates of the meet of two given planes in terms of the coordinates of these planes.

3. The *shortest distance* between the lines $s_1(\lambda_1, \mu_1, \nu_1, p_1, q_1, r_1)$, $s_2(\lambda_2, \mu_2, \nu_2, p_2, q_2, r_2)$ is

$$\frac{\lambda_1 p_2+\mu_1 q_2+\nu_1 r_2+\lambda_2 p_1+\mu_2 q_1+\nu_2 r_1}{\{(\mu_1\nu_2-\mu_2\nu_1)^2+(\nu_1\lambda_2-\nu_2\lambda_1)^2+(\lambda_1\mu_2-\lambda_2\mu_1)^2\}^{1/2}}.$$

$s_1$, $s_2$ *intersect if and only if*

$$\lambda_1 p_2+\mu_1 q_2+\nu_1 r_2+\lambda_2 p_1+\mu_2 q_1+\nu_2 r_1=0.$$

[Expand the determinant in **10** (4) using the definitions (3).]

* Appeal to a mechanical analogy clarifies the significance of line-coordinates; see Salmon, *Analytic Geometry of Three Dimensions*, § 53.

## CHAPTER III

# SPHERE

THIS chapter is inserted in order to give elementary illustrations of the use of the results already found, and to provide some introduction to later general theory, of which it gives particular cases.

### 14. Sphere referred to its Centre

Let $S$ be a sphere of radius $a$. Take axes through origin $O$ at the centre of $S$. Then $P(x, y, z)$ is on $S$ if and only if $|OP| = a$, or

$$x^2+y^2+z^2 = a^2, \quad . \quad . \quad . \qquad (1)$$

which is therefore the equation of $S$.

Let $Q(x', y', z')$ be a fixed point, and consider the line $s$ through $Q$ with direction $(l, m, n)$. Any point $P(x, y, z)$ of $s$ is given by (**11** (1)) $x = x'+lr$, $y = y'+mr$, $z = z'+nr$, where $r = (QP)$. $P$ lies also on $S$ if

$$(x'+lr)^2+(y'+mr)^2+(z'+nr)^2 = a^2,$$

*i.e.*

$$r^2+2(lx'+my'+nz')r+x'^2+y'^2+z'^2-a^2 = 0. \qquad (2)$$

This is a quadratic equation for $r$ having roots $r_1$, $r_2$ (say), both real or both complex. If $r_1$, $r_2$ are real and distinct, $s$ meets $S$ in two points $P_1$, $P_2$ (say). If $r_1 = r_2$ (then necessarily real), $s$ meets $S$ in one point $P_1$ which may be regarded as given by the coincidence of $P_1, P_2$; we say that *$s$ touches $S$*, or is tangent to $S$, with *point of contact* $P_1$. If $r_1, r_2$ are not real, $s$ does not meet $S$.

From (2) we have

$$(QP_1).(QP_2) \equiv r_1 r_2 = x'^2+y'^2+z'^2-a^2. \tag{3}$$

Hence *if any line through a point $Q$ meets a sphere $S$ in points $P_1$, $P_2$ the product $(QP_1).(QP_2)$ depends only on $Q$, $S$.* We call this the *power* $p(Q\,;S)$ of $Q$ w.r.t. $S$.

Equation (3) may be interpreted as

$$p(Q\,;S) = (\text{distance of } Q \text{ from centre of } S)^2-(\text{radius of } S)^2.$$

Therefore $p(Q\,;S)$ is (i) $< 0$, if $Q$ is inside $S$, and (3) shows that $Q$ then lies between $P_1$, $P_2$; (ii) $= 0$, if $Q$ is on $S$; (iii) $> 0$, if $Q$ is outside $S$, and (3) shows that $Q$ is then not between $P_1$, $P_2$. If $r_1 = r_2$, (3) gives

$$p(Q\,;S) = (\text{length of tangent from } Q \text{ to } S)^2 \geqslant 0,$$

and $Q$ is then not inside $S$.

Again from (2) we have

$$(QP_1)+(QP_2) \equiv r_1+r_2 = -2(lx'+my'+nz'). \tag{4}$$

$Q$ is the midpoint of $(P_1P_2)$ if $(QP_1) = -(QP_2)$, *i.e.* from (4), if $lx'+my'+nz' = 0$. Therefore, if $l$, $m$, $n$ are fixed, $Q$ must lie in the plane

$$lx+my+nz = 0.$$

This contains $O$ and is normal to $(l, m, n)$. Hence *the midpoints of chords of a sphere parallel to a fixed direction lie in the diametral plane normal to the latter.*

Now let $Q$ be on $S$, so that $x'^2+y'^2+z'^2 = a^2$. Then the roots of (2) become $r_1 = 0$, $r_2 = -2(lx'+my'+nz')$. The vanishing of $r_1$ means merely that every line through $Q$ meets $S$ in $Q$, *i.e.* $P_1 \equiv Q$. Then $P_2 \equiv P_1$, making $s$ tangent to $S$ at $Q$, if and only if $r_2 = 0$, giving

$$lx'+my'+nz' = 0.$$

This expresses that $(l, m, n)$ is perpendicular to $OQ$. Hence a line through a point $Q$ on $S$ is tangent to $S$ (at $Q$) if and only if it is perpendicular to the radius through $Q$. So

*every tangent line at $Q$ lies in the plane through $Q$ normal to $OQ$, and every line through $Q$ in this plane is a tangent line at $Q$;* this is called the *tangent plane* at $Q$. These properties show that its equation is

$$x'(x-x')+y'(y-y')+z'(z-z') = 0,$$

or, since $x'^2+y'^2+z'^2 = a^2$,

$$xx'+yy'+zz' = a^2. \qquad (5)$$

Suppose now that (5) passes through a fixed point $R(x'', y'', z'')$. Then $x'x''+y'y''+z'z'' = a^2$, showing that $Q$ then lies in the fixed plane

$$xx''+yy''+zz'' = a^2. \qquad (6)$$

Hence *the points of contact of tangent lines to $S$ from a fixed point $R$ lie in a fixed plane.* This is called the *plane of contact* of $R$ and the locus of these tangent lines the *tangent cone* from $R$.

It is clear that the foregoing geometrical theorems, which are trivial in themselves and are reproduced merely to illustrate the algebraic method, are independent of the particular choice of axes used in deducing them. The *form* of their algebraic expression, on the other hand, does depend on this choice.

1. Tangent lines of $S$ pass through a given point $R$ if and only if $R$ is not inside $S$. [For the plane (6) meets $S$ if and only if its perpendicular distance from $O$ is not greater than $a$, *i.e.* $x''^2+y''^2+z''^2-a^2 \geqslant 0$.]

2. *Polar plane.* Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be such that *the sum of their powers w.r.t. $S$ is equal to the square of their mutual distance.* Then

$$x_1x_2+y_1y_2+z_1z_2 = a^2.$$

If $P_1$ is fixed, $P_2$ must therefore lie in the plane

$$xx_1+yy_1+zz_1 = a^2, \qquad (7)$$

and conversely, if $P_2$ is *any* point of (7), $P_1$, $P_2$ are related in this way. [(7) is called the *polar plane* of $P_1$ w.r.t. $S$. This is

not a standard definition, but most such definitions fail to apply generally to every position of $P_2$ in the polar plane.]

3. Using (7), establish the properties of pole and polar in **33** for the particular case of a sphere.

## 15. General Equation of a Sphere

The sphere with centre $(\alpha, \beta, \gamma)$ and radius $a$ is seen to be

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 = a^2. \qquad (1)$$

This equation has the form

$$cx^2+cy^2+cz^2+2ux+2vy+2wz+d = 0, \qquad (2)$$

where $c$, $d$, $u$, $v$, $w$ are constants ($c \neq 0$). Conversely, given any equation of the form (2), we may write it

$$\left.\begin{aligned} (x+u/c)^2+(y+v/c)^2+(z+w/c)^2 &= k, \\ \text{where} \quad k &= (u^2+v^2+w^2-cd)/c^2. \end{aligned}\right\} \qquad (3)$$

Comparing (3) with (1), we see that its locus is the sphere having centre $C(-u/c,\ -v/c,\ -w/c)$ and radius $\sqrt{k}$, provided $k \geqslant 0$. The case $k = 0$ gives a sphere of zero radius (point-sphere) reducing to the single point $C$. There are no points which satisfy (3) if $k < 0$. Hence *the equation of every sphere is of the form* (2), *and the locus, when it exists, of every such equation is a sphere.* We usually assume (2) to be "normalised" so that $c = 1$.

1. The general equation of a sphere depends on *four* independent constants. One and only one sphere passes through four given non-coplanar points.

By the methods of **14**, or by merely changing the origin in the results of **14**, we prove: $S$ being given by (2) with $c = 1$, and $Q$ being $(x', y', z')$, the power of $Q$ is given by

$$p(Q\,;\,S) = x'^2+y'^2+z'^2+2ux'+2vy'+2wz'+d; \qquad (4)$$

and the equation

$$xx'+yy'+zz'+u(x+x')+v(y+y')+w(z+z')+d = 0$$

gives the *tangent plane* at $Q$ if $Q$ is on $S$, the *plane of contact* of $Q$ if $Q$ is outside $S$, the *polar plane* of $Q$ for all $Q$.

## 16. Circle

If a plane and a sphere intersect, they do so in a circle. So the locus of

$$\left.\begin{aligned} a'x+b'y+c'z+d' &= 0, \\ x^2+y^2+z^2+2ux+2vy+2wz+d &= 0, \end{aligned}\right\} \quad . \qquad (1)$$

taken together, when it exists, is a circle. Conversely, every circle can be represented by equations of these forms.

As in the case of a line, more than one equation is needed to specify a circle in $\mathcal{E}$. We cannot specify a circle more simply than by one equation of the first degree and one of the second. In this specification the former equation gives the unique plane containing the circle, but the latter need not be that of a sphere; it may be a quadric of any type possessing circular sections (**44**).

1. Find the condition for the circle (1) to exist, and find its centre. [The distance of the plane from the centre of the sphere must not exceed its radius.]

## 17. Radical Plane

Let

$$\begin{aligned} S_1 &\equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = 0, \\ S_2 &\equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2 = 0, \end{aligned}$$

be two given spheres, and $Q(x', y', z')$ any point. Then $p(Q\,;S_1) = p(Q\,;S_2)$ if and only if, using **15** (4),

$$\begin{aligned} x'^2+y'^2+z'^2+2u_1z'+2v_1y'+2w_1z'+d_1 \\ = x'^2+y'^2+z'^2+2u_2z'+2v_2y'+2w_2z'+d_2, \end{aligned}$$

*i.e.* if and only if $Q$ lies in the plane

$$2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2 = 0. \quad (1)$$

This is normal to the direction $(u_1-u_2, v_1-v_2, w_1-w_2)$ and so to the line of centres of $S_1$, $S_2$. Also, if $R$ is any point common to $S_1$, $S_2$, then $p(R\,;S_1) = 0 = p(R\,;S_2)$; so, if $S_1$, $S_2$ intersect, their intersection lies in (1). Hence *the locus of a point having equal powers w.r.t. two spheres is a plane normal to their line of centres and containing their intersection, if any.* It is called their *radical plane.*

1. Let $S_i = 0$ $(i = 1, 2, 3)$ be the normalised equations of three given spheres. The three radical planes of $S_1$, $S_2$, $S_3$ taken in pairs meet in a line [given by $S_1 = S_2 = S_3$, and sometimes called their *radical axis*], or are parallel.

2. Let $S_i = 0$ $(i = 1, 2, 3, 4)$ be the normalised equations of four given spheres. The six radical planes of $S_1$, $S_2$, $S_3$, $S_4$ have in general a point in common [given by $S_1 = S_2 = S_3 = S_4$, and called the *radical centre.* Consider cases of exception].

## 18. Coaxial Spheres

A sphere $S$ and a plane $\Pi$ being given by **16** (1), consider the equation

$$S_\lambda(x, y, z) \equiv S+2\lambda\Pi \equiv x^2+y^2+z^2+2(u+\lambda a')x$$
$$+2(v+\lambda b')y+2(w+\lambda c')z+d+2\lambda d' = 0,$$

where $\lambda$ is any constant. It is the equation of a sphere $S_\lambda$. Let $Q(x', y', z')$ be any point of $\Pi$ so that $\Pi(x,' y,' z') = 0$. Then (**15** (4))

$$p(Q\,;S_\lambda) = S_\lambda(x', y', z') = S(x', y', z')+2\lambda\Pi(x', y', z')$$
$$= S(x', y', z') \equiv q, \quad \text{say,}$$

where $q$ depends on $Q$ but not on $\lambda$. Hence as $\lambda$ varies the spheres $S_\lambda$ form a family such that the power of $Q$ w.r.t. every member is the same, *i.e.* $\Pi$ is the radical plane of every pair of members. It follows that the spheres have also a

common line of centres. They are said to constitute a *coaxial system*.

1. Give the coordinates of the centre of $S_\lambda$; verify that it lies on a fixed normal to $\Pi$.

2. There is one and only one sphere $S_\lambda$ through any point not on $\Pi$. Either no two spheres of the system intersect, or else they all contain a common circle in $\Pi$.

3. If any line $s$ through $Q$, but not contained in $\Pi$, meets $S_\lambda$ in $P'_\lambda$, $P''_\lambda$, then $(QP'_\lambda)\,.\,(QP''_\lambda) = q$; hence, as $\lambda$ varies, $P'_\lambda$, $P''_\lambda$ generate an *involution* on $s$. If $Q$ is outside the spheres, *two* of them touch $s$; if $Q$ is inside the spheres, none touches $S$.

4. There are two point-spheres (*limiting points*) of a non-intersecting coaxial system.

5. *Orthogonal spheres.* Spheres

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0,$$
$$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0,$$

intersect orthogonally at every common point if and only if $2uu'+2vv'+2ww' = d+d'$. [This is a simple extension of the corresponding result for circles in a plane.]

6. If a sphere $S$ is orthogonal to each of two spheres $S_1$, $S_2$, its centre $C$ lies in the radical plane $\Pi$ of $S_1$, $S_2$. Conversely, if $C$ lies in $\Pi$, and if $S$ is orthogonal to $S_1$, it is orthogonal to $S_2$ and so to every member of the coaxial system determined by $S_1$, $\Pi$.

7. All the spheres orthogonal to three given spheres form in general a coaxial system whose radical plane is the plane of centres of the given spheres and whose line of centres is their radical axis.

8. There is in general one and only one sphere orthogonal to four given spheres.

## 19. Parametric Representation

Let $\Pi^*$ be a fixed plane and let $u$, $v$ be cartesian coordinates of any point $P^*$ of $\Pi^*$ referred to axes in $\Pi^*$ itself. We saw (**11** 6) that the parametric representation of any other plane $\Pi$ yields a "map" of $\Pi$ on $\Pi^*$. Conversely,

*if* we can map any surface $\Sigma$ on $\Pi^*$ we shall obtain a parametric representation of $\Sigma$. For, if $P(x, y, z)$ is any point of $\Sigma$ and $P^*(u, v)$ its "map" in $\Pi^*$, then $x, y, z$ must be expressible in terms of $u, v$.

One way of mapping a sphere $S$ (centre $O$, radius $a$) on $\Pi^*$ is by *stereographic projection*, the simplest case being the following: Take as $\Pi^*$ any plane through $O$. Let $A$ be one extremity of the diameter normal to $\Pi^*$; then the join of $A$ to any other point $P$ of $S$ meets $\Pi$ in one point $P^*$, and conversely. This gives a one-to-one correspondence between $P, P^*$; so $P^*$ may be taken as the "map" of $P$.

Now take axes with origin $O$ and **OZ** along **OA**. Then $\Pi^*$ is the plane $z = 0$; $A$ is the point $(0, 0, a)$; $S$ is the sphere

$$x^2+y^2+z^2 = a^2. \qquad (1)$$

Let $P^*$ be the point $(u, v, 0)$; so $AP^*$ has parametric equations

$$x = ut, \quad y = vt, \quad z = a(1-t). \qquad (2)$$

This meets $S$ at points where, from (1),

$$u^2t^2+v^2t^2+a^2(1-t)^2 = a^2,$$

whence $t = 0$ or $2a^2/(u^2+v^2+a^2)$. The first root gives $A$; therefore the second must give $P$. Substituting for $t$ in (2) the coordinates of $P$ are

$$x = \frac{2a^2u}{u^2+v^2+a^2}, \quad y = \frac{2a^2v}{u^2+v^2+a^2}, \quad z = a\frac{u^2+v^2-a^2}{u^2+v^2+a^2}. \qquad (3)$$

Also (2) gives

$$u = ax/(a-z), \qquad v = ay/(a-z). \qquad (4)$$

(3) is a *parametric representation* of $S$. (3), (4) render explicit the (1–1) correspondence between $P, P^*$; if $u, v$ are given, (3) determines unique values of $x, y, z$ satisfying (1); if $x, y, z$ satisfying (1) are given, (4) determines unique values of $u, v$, with the single exception that no point of $\Pi^*$ corresponds to $A$ on $S$.

1. The curves $u =$ constant, $v =$ constant, *on* $S$ are two families of small circles through $A$, one member of each going through every point of $S$ other than $A$.

2. A more familiar method of locating a point on a sphere is by its latitude and longitude. The following slight modification of this is used in mathematics: In the frame of reference used above, call the angle $ZOP$ the *colatitude* $\theta$ of $P$, and the angle between the planes $OZX$, $OZP$, in the r.h sense of rotation about $OZ$, the *azimuth* $\phi$ of $P$. Then, if $P$ is given, $\theta$, $\phi$ are uniquely determined so that $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$; if any pair of values of $\theta$, $\phi$ in this range is given, it locates a unique point $P$ on $S$. Hence $\theta$, $\phi$ serve as parameters of $P$. Projecting $\overline{OP}$ on the axes we obtain the parametric equations

$$x = a \sin\theta \cos\phi, \quad y = a \sin\theta \sin\phi, \quad z = a\cos\theta. \qquad (5)$$

This non-algebraic representation is not so useful as (3) in algebraic geometry.

3. *Spherical polar coordinates.* Let $Q(x, y, z)$ be *any* point of $\mathcal{E}$. Let $P$ be either of the two points in which $OQ$ meets $S$, and let $P$ be specified by $\theta$, $\phi$ as in 2. Taking $\overline{OP}$ as fixing the positive sense of $OQ$, let $(OQ) = r$. Then as in (5) we obtain

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Thus $Q$ is determined uniquely when $r$, $\theta$, $\phi$ are given; these are called *spherical polar coordinates* of $Q$ in the frame of reference used. Note however that, if $Q$ is given, $r$, $\theta$, $\phi$ are not uniquely determined, owing to the ambiguity in the choice of $P$.

4. If in 3 we take $r\sin\theta = \rho$, $Q$ is determined uniquely when $\rho$, $\phi$, $z$ are given; these are called *cylindrical polar coordinates* of $Q$.

CHAPTER IV

# HOMOGENEOUS COORDINATES—POINTS AT INFINITY

THE preceding work suffers from certain inelegancies of form and expression: of form, since we specify any point by a unique set of *three* point-coordinates, and go on to specify any plane by a non-unique set of *four* plane-coordinates, and any line by a set of line-coordinates of *non-uniform* type; of expression, since we have to make continual mention of special cases depending on *parallelism.* The simple algebraic device now to be introduced removes these defects and simplifies the subsequent algebra. It does *not* mean that we pass to a different geometry. In particular, we do not discard the special cases due to parallelism; they merely cease to be exceptional as regards the general treatment, but can be isolated whenever necessary. This is well illustrated by the successive stages in the classification of quadrics (Ch. V).

## 20. Homogeneous Coordinates

Suppose we have a collection $\mathcal{G}$ of geometrical objects and we label them with sets of numbers $(g_1, g_2, \ldots, g_n)$. If (i) to every set of values of $g_1, \ldots, g_n$ in a stated range there corresponds a unique member of $\mathcal{G}$, which we call "the object $(g_1, \ldots, g_n)$," (ii) the objects $(g_1', \ldots, g_n')$, $(g_1'', \ldots, g_n'')$ are the same if and only if $g_1'/g_1'' = \ldots = g_n'/g_n''$, (iii) this system of labelling includes all the members of $\mathcal{G}$, then we call $g_1, \ldots, g_n$ homogeneous coordinates of the corresponding object in $\mathcal{G}$.

Since the set of values 0, 0, ..., 0 could be reckoned proportional to any other set, it must be excluded from the range of possible values of $g_1, \ldots, g_n$.

Now let some geometrical relation concerning the object $(g_1', \ldots, g_n')$ be expressed by an algebraic equation involving these coordinates. Then that equation must be satisfied when these coordinates are replaced by the "equivalent" set $(g_1'', \ldots, g_n'')$, where $g_1'/g_1'' = \ldots = g_n'/g_n''$, since this set specifies the same geometrical object. Hence the equation must be *homogeneous* in $g_1', \ldots, g_n'$.

## 21. Linear Dependence

We denote a set $(g_1, \ldots, g_n)$ by the symbol $\mathbf{g}$, and a set $(kg_1, \ldots, kg_n)$ by $k\mathbf{g}$. We can without confusion denote also by $\mathbf{g}$ the geometrical object corresponding to the set $\mathbf{g}$; then $\mathbf{g}$ and $k\mathbf{g}$ denote the same object $(k \neq 0)$.

Sets $\mathbf{g}', \mathbf{g}'', \ldots, \mathbf{g}^{(r)}$ are said to be *linearly dependent* if there exist numbers $k', k'', \ldots, k^{(r)}$, not all zero, such that

$$k'\mathbf{g}'+k''\mathbf{g}''+\ldots+k^{(r)}\mathbf{g}^{(r)} = 0, \quad . \tag{1}$$

where this implies the $n$ equations

$$k'g_i'+k''g_i''+\ldots+k^{(r)}g_i^{(r)} = 0. \quad (i = 1, \ldots, n) \tag{2}$$

If in particular $k^{(p)}$, say, $(1 \leqslant p \leqslant r)$ is not zero, we say that $\mathbf{g}^{(p)}$ is linearly dependent on the remaining $\mathbf{g}$'s in (1). Sets which are not linearly dependent are called linearly independent. We require the following lemmas:

(i) There exists a $\mathbf{g}$ linearly dependent on $\mathbf{u}, \mathbf{v}, \ldots$, and on $\mathbf{y}, \mathbf{z}, \ldots$, if and only if $\mathbf{u}, \mathbf{v}, \ldots, \mathbf{y}, \mathbf{z}, \ldots$ are linearly dependent amongst themselves. This follows from the definition.

(ii) Any $m$ of the $\mathbf{g}$'s are linearly dependent if $m > n$ (Aitken, **28** 3).

(iii) At least *two* of the $k$'s in (1) are not zero. For if all the $k$'s except $k^{(p)}$ are zero, (2) reduces to $k^{(p)}g_i^{(p)} = 0$ $(i = 1, \ldots, n)$.

But we have excluded the possibility $g_1^{(p)} = \ldots = g_n^{(p)} = 0$. Hence $k^{(p)} = 0$, and we have a contradiction.

(iv) Two geometrical objects $\mathbf{g}'$, $\mathbf{g}''$ are identical if and only if $\mathbf{g}'$, $\mathbf{g}''$ are linearly dependent. For, if $k'\mathbf{g}'+k''\mathbf{g}'' = 0$, then, by (iii), $k'$, $k'' \neq 0$, so $g_1'/g_1'' = \ldots = g_n'/g_n''$, and conversely. Bearing in mind this lemma, we may speak of linear dependence of sets of coordinates in terms of the corresponding geometrical objects.

## 22. Point-coordinates : Space $\overline{\mathscr{E}}$

Let $P(x, y, z)$ be any point of $\mathscr{E}$. We shall now label $P$ by *four* numbers $x_1$, $x_2$, $x_3$, $x_4$ such that

$$x = x_1/x_4, \quad y = x_2/x_4, \quad z = x_3/x_4. \qquad (1)$$

If, in the definition in **20**, $\mathscr{G}$ is the aggregate of points of $\mathscr{E}$ and $x_1, \ldots, x_4$ can take all values, then we see that $x_1$, $x_2$, $x_3$, $x_4$ are homogeneous coordinates of $P$ in $\mathscr{E}$, *provided we exclude the possibility* $x_4 = 0$.

Suppose now we *define* a collection $\overline{\mathscr{E}}$ of objects as those having homogeneous coordinates $x_1$, $x_2$, $x_3$, $x_4$, where these range unrestrictedly over all real values (except only the set 0, 0, 0, 0). We shall, as is possible from what has just been said, identify those objects of $\overline{\mathscr{E}}$ for which $x_4 \neq 0$ with the points of $\mathscr{E}$. We shall then choose to call all the objects of $\overline{\mathscr{E}}$ the "points" of $\overline{\mathscr{E}}$. When it is necessary to make the distinction, we shall call points having $x_4 \neq 0$ "ordinary" points of $\overline{\mathscr{E}}$, and those having $x_4 = 0$ "special" points. The latter are not points of $\mathscr{E}$; so we have hitherto no geometrical definitions concerning them and we are consequently now free to impose any we please (provided they are self-consistent).

**Reversion to cartesian coordinates.** If $\mathbf{x}$ is any ordinary point $P$, then, $k\mathbf{x}$ being the same point for all $k$ and $x_4$ being not zero, we can take $k = 1/x_4$ and so obtain for $P$, from (1), the particular set of coordinates $(x, y, z, 1)$. We can thus make the $x_1$, $x_2$, $x_3$-coordinates equal to the

original cartesian coordinates. Further, since we deal only with *homogeneous* equations in $x_1$, $x_2$, $x_3$, $x_4$, these equations hold good equally for every particular set of homogeneous coordinates of $P$, in particular for that in which $x_4 = 1$. Therefore, in order to pass to the cartesian form of an equation, we have merely to put $x_4 = 1$ and replace $x_1$, $x_2$, $x_3$ by $x$, $y$, $z$.

**Notation.** When using homogeneous point-coordinates we reserve subscripts to distinguish the four coordinates of the same point. To distinguish different points we use either superscripts or else different letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{t}$, ... ($\mathbf{x}$ generally denoting a variable point). The practice is largely the reverse of that followed with cartesian coordinates, and the reader should guard against confusion. Also we formerly spoke of "the point $P(x, y, z)$," specifying it descriptively by $P$ and algebraically by $(x, y, z)$; now we can make a single symbol like $\mathbf{x}$ do the double duty. Analogous remarks apply to the subsequent treatment of planes.

## 23. Linear Dependence of Points

(*a*) The relevant objects being points of $\overline{\mathscr{E}}$, lemma (iv) becomes:

A. *If two points are linearly dependent they are coincident, and conversely.*

(*b*) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be the ordinary points $P_1$, $P_2$, $P_3$ of **6**. Writing $\lambda = \mu_1/x_4$, etc., the four equations **6** (2) may now be replaced by the single condition

$$\lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w} = 0. \qquad (1)$$

Hence the theorem in **6** is simply: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are collinear if and only if they are linearly dependent. Therefore, $\mathbf{u}$, $\mathbf{v}$ being distinct ordinary points, their join $s$ in $\mathscr{E}$ consists of all ordinary points $\mathbf{x}$ given by

$$\mathbf{x} = \lambda\mathbf{u}+\mu\mathbf{v}, \qquad (2)$$

where $\lambda$, $\mu$ take all real values (not both zero). There is one and only one distinct point of $s$ corresponding to every value of the ratio $\lambda : \mu$, except $\lambda : \mu = -v_4 : u_4$, which makes $x_4 = 0$. The latter gives a unique special point in $\bar{\mathscr{E}}$. We shall now call the aggregate of *all* points given by (2) an *ordinary line* $\bar{s}$ in $\bar{\mathscr{E}}$; so $\bar{s}$ is identical with $s$, except for the addition of one special point.

1. In (2) we may replace $\mathbf{v}$, say, by the special point of $\bar{s}$.

Now let $\mathbf{u}$, $\mathbf{v}$ be distinct special points. Then $u_4 = v_4 = 0$, and so $x_4 = 0$ for all $\lambda$, $\mu$ in (2). All points given by (2) are then special, and we call their aggregate a *special line* in $\bar{\mathscr{E}}$.

*Line* will now mean either ordinary or special line. We see that the definitions extend to $\bar{\mathscr{E}}$ the fundamental property of a line in $\mathscr{E}$, that it is completely determined by any two distinct points lying on it. They permit the assertion regarding $\bar{\mathscr{E}}$:

B. *If three points are linearly dependent they are collinear, and conversely.*

2. If a line contains one ordinary point, it is ordinary.

3. The point $\mathbf{x}$ given by (2) divides the join of $\mathbf{u}$, $\mathbf{v}$ in the ratio $\mu v_4 : \lambda u_4$.

4. If two lines are given by $\mathbf{x} = \lambda\mathbf{u}+\mu\mathbf{v}$, $\mathbf{x} = \nu\mathbf{w}+\varpi\mathbf{z}$, they have a common point if and only if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ are linearly dependent. [Lemma (i).]

(*c*) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ be the ordinary points $P_1$, $P_2$, $P_3$, $P_4$ of **7**. As before, the four equations **7** (1) can now be replaced by the single condition

$$\lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w}+\varpi\mathbf{z} = 0. \qquad (3)$$

Hence the theorem in **7** is: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ are coplanar if and only if they are linearly dependent. Therefore, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ being distinct non-collinear points, their plane $\Pi$ in $\mathscr{E}$ consists of all ordinary points $\mathbf{x}$ given by

$$\mathbf{x} = \lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w}, \qquad (4)$$

where $\lambda$, $\mu$, $\nu$ take all real values (not all zero). There is one and only one distinct point of $\Pi$ corresponding to every value of the ratios $\lambda : \mu : \nu$, except those satisfying

$$\lambda u_4 + \mu v_4 + \nu w_4 = 0, \qquad . \qquad . \qquad (5)$$

which make $x_4 = 0$. The latter give a unique special line in $\overline{\mathscr{E}}$. For all points satisfying (4) and (5) give

$$w_4 \mathbf{x} = \lambda(w_4 \mathbf{u} - u_4 \mathbf{w}) + \mu(w_4 \mathbf{v} - v_4 \mathbf{w}), \qquad . \qquad (6)$$

and conversely. From (*b*), (6) gives the special line determined by the special points $w_4\mathbf{u} - u_4\mathbf{w}$, $w_4\mathbf{v} - v_4\mathbf{w}$, which are distinct since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent. We shall now call the aggregate of *all* points given by (4) an *ordinary plane* $\overline{\Pi}$ in $\overline{\mathscr{E}}$; so $\overline{\Pi}$ is identical with $\Pi$, except for the addition of one special line.

5. In (4) we may replace $\mathbf{v}$, $\mathbf{w}$, say, by any two points of the special line in $\overline{\Pi}$.

Now let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be non-collinear special points. All the points given by (4) are then special, and we call their aggregate a *special plane* in $\overline{\mathscr{E}}$.

*Plane* will now mean either ordinary or special plane. The definitions extend to $\overline{\mathscr{E}}$ the fundamental property of a plane in $\mathscr{E}$, that it is completely determined by any three non-collinear points lying in it. They permit us to assert regarding $\overline{\mathscr{E}}$:

C. *If four points are linearly dependent they are coplanar, and conversely.*

6. If a plane contains one ordinary point, it is ordinary.

7. A plane in $\overline{\mathscr{E}}$ entirely contains the line determined by any two of its points.

8. Every line in a special plane is special.

9. $\lambda$, $\mu$, $\nu$ in (4) provide homogeneous coordinates of the points of the plane $\mathbf{uvw}$. [When $u_4 = v_4 = w_4 = 1$, they are areal, or barycentric, coordinates with $\mathbf{uvw}$ as triangle of reference.]

(*d*) Since a point of $\bar{\mathcal{E}}$ is given by *four* coordinates, it follows from lemma (ii) that any five points are linearly dependent, and that all points $\mathbf{x}$ of $\bar{\mathcal{E}}$ are given by

$$\mathbf{x} = \lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w}+\varpi\mathbf{z}, \qquad (7)$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ are any four non-coplanar points and $\lambda$, $\mu$, $\nu$, $\varpi$ take all real values (not all zero).

Just as we proved in (*c*) that every special point of (4) lies in a unique special line, it now follows that every special point of (7) lies in a unique special plane. So *there is one and only one special plane* in $\bar{\mathcal{E}}$, and $\bar{\mathcal{E}}$ is identical with $\mathcal{E}$, except for the addition of this special plane.

(7) permits us to assert regarding $\bar{\mathcal{E}}$:

D. *Every point is linearly dependent on any four linearly independent points.*

10. The numbers $\lambda$, $\mu$, $\nu$, $\varpi$ in (7) provide a new set of homogeneous coordinates of the point $\mathbf{x}$. When $u_4 = v_4 = w_4 = z_4 = 1$, they are called *barycentric* coordinates with tetrahedron of reference having vertices $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$. The equation of the special plane in barycentric coordinates is $\lambda+\mu+\nu+\varpi = 0$.

We now have the following incidence relations in $\bar{\mathcal{E}}$ (in addition to 4, 7):

(i) *Every line $s$ meets every plane $\Pi$ in which it does not lie in one and only one point.** Let $\mathbf{y}$, $\mathbf{z}$ be distinct points of $s$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ non-collinear points of $\Pi$. Then by (*d*) there exist $\rho, \ldots, \nu$ (no four being zero, lemma (iii)) such that

$$\rho\mathbf{y}+\varpi\mathbf{z}+\lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w} = 0.$$

Also $\rho$, $\varpi$ are not both zero, since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent; $\lambda$, $\mu$, $\nu$ are not all zero, since $\mathbf{y}$, $\mathbf{z}$ are distinct. So there exists a point $\mathbf{x}$ such that

$$\mathbf{x} = \rho\mathbf{y}+\varpi\mathbf{z} = -(\lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w}).$$

* Henceforth we use, when necessary, symbols like $s$, $\Pi$ (without the bar) for a line, plane in $\bar{\mathcal{E}}$.

Using (*b*), (*c*) it follows that $\mathbf{x}$ lies in both $s$ and $\Pi$. From 7, $\mathbf{x}$ is unique, for if $s$ contains two points of $\Pi$ it lies in $\Pi$.

(ii) *Every two distinct planes* $\Pi$, $\Lambda$ *have one and only one line in common.* Take any two distinct lines in $\Pi$ through a point of $\Pi$ not in $\Lambda$. These meet $\Lambda$ in unique points (by (i)) whose join $s$ lies in $\Pi$, $\Lambda$ (by **7**). $\Pi$, $\Lambda$ cannot have any common point not on $s$; otherwise (by (*c*)) they would coincide.

(iii) *Every three non-collinear planes have one and only one common point.* (Planes are called "collinear" if they have a line in common, "concurrent" if they have a point in common.) This follows from (i), (ii).

Note that in $\overline{\mathscr{E}}$ there are no exceptions to these relations due to parallelism.

## 24. Plane-coordinates: Linear Dependence of Planes

Either directly from results in **8**, or by using the methods of **8** in conjunction with **23** (*c*), we now have: *The equation of every plane is a homogeneous linear equation in homogeneous point-coordinates; the locus of every such equation is a plane.* We write such an equation in the form

$$\boldsymbol{\xi}\,.\,\mathbf{x} \equiv \xi_1x_1+\xi_2x_2+\xi_3x_3+\xi_4x_4 = 0, \qquad (1)$$

where $\mathbf{x} \equiv (x_1,\ x_2,\ x_3,\ x_4)$ is the variable point, and the coefficients $\xi_1$, $\xi_2$, $\xi_3$, $\xi_4$ are not all zero.

From **12**, **20** we see that $\xi_1$, $\xi_2$, $\xi_3$, $\xi_4$ are homogeneous coordinates of the plane (1), which we now call the plane $\boldsymbol{\xi}$. In $\mathscr{E}$ the case $\xi_1 = \xi_2 = \xi_3 = 0$ was excluded; but this now gives, from (1), $x_4 = 0$, which is just the equation of the special plane in $\overline{\mathscr{E}}$.

Using the methods and results of **9**, **23** we can deduce the following (for which the reader should construct proofs):

A$'$. *If two planes are linearly dependent they are coincident, and conversely.*

B$'$. *If three planes are linearly dependent they are collinear, and conversely.*

C′. *If four planes are linearly dependent they are concurrent, and conversely.*

D′. *Every plane is linearly dependent on any four linearly independent planes.*

## 25. Duality

We now observe that, if in any one of the relations **23** A–D we replace the terms

point line plane

by the terms

plane line point,

respectively, we obtain the corresponding one of the relations **24** A′–D′, and *vice versa*. Since any incidence relations derived subsequently will depend only on these fundamental ones, it will remain true that when this replacement is made in any such relation a valid relation will result. Pairs of relations corresponding in this manner are called *dual* relations, and the principle which allows us to pass from one to the other is called the *principle of duality*.

## 26. Line-coordinates

Let $s$ be a line and $\mathbf{y}$, $\mathbf{z}$ any two points of $s$. Then, as in **6** (4), $\mathbf{x}$ lies in $s$ if and only if

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix} = 0. \tag{1}$$

By **6** 3, a necessary and sufficient condition for (1) is given by any two distinct equations of the following set, from which the rest follow :

$$\begin{aligned} p_{23}x_1+p_{31}x_2+p_{12}x_3 &= 0, \\ p_{24}x_1+p_{41}x_2+p_{12}x_4 &= 0, \\ p_{34}x_1+p_{41}x_3+p_{13}x_4 &= 0, \\ p_{34}x_2+p_{42}x_3+p_{23}x_4 &= 0, \end{aligned} \tag{2}$$

where $$p_{12} \equiv y_1z_2-y_2z_1 \equiv -p_{21}, \text{ etc.} \tag{3}$$

The ratios of $p_{12}, p_{13}, \ldots$ are fixed when the points $\mathbf{y}, \mathbf{z}$ are given; not all of $p_{12}, p_{13}, \ldots$ are zero, since $\mathbf{y}, \mathbf{z}$ are distinct points. Also by expanding the vanishing determinant

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

on the first two rows, we have the identity

$$p_{12}p_{34}+p_{23}p_{14}+p_{13}p_{42} = 0. \tag{4}$$

Conversely, as in **13**, the planes (2) all pass through a unique line $s$ when the ratios of $p_{12}, p_{13}, \ldots$ satisfying (4) are given. Hence $p_{12}, p_{13}, \ldots$ related by (4) provide homogeneous coordinates for the lines of $\overline{\mathscr{E}}$. They are known as *Plücker coordinates.*

We can derive a second set of coordinates $\varpi_{12}, \varpi_{13}, \ldots$ by taking the dual of every step in the foregoing derivation. In particular, if $\boldsymbol{\eta}, \boldsymbol{\zeta}$ are any two planes through $s$ we take, dually to (3),

$$\varpi_{12} \equiv \eta_1\zeta_2-\eta_2\zeta_1 \equiv -\varpi_{21}, \text{ etc.} \tag{5}$$

But the first two planes (2) could be taken for $\boldsymbol{\eta}, \boldsymbol{\zeta}$, which are then the planes $(p_{23}, p_{31}, p_{12}, 0)$, $(p_{24}, p_{41}, 0, p_{12})$. Substituting in (5) and using (4), we find

$$\varpi_{12} : \varpi_{13} : \varpi_{14} : \varpi_{23} : \varpi_{24} : \varpi_{34} = p_{34} : p_{42} : p_{23} : p_{14} : p_{31} : p_{12}. \tag{6}$$

1. Verify that passage to cartesian coordinates gives the results of **13** 1, 2.

2. The lines $\mathbf{p}, \mathbf{q}$ intersect if and only if

$$p_{12}q_{34}+p_{13}q_{42}+p_{14}q_{23}+p_{23}q_{14}+p_{24}q_{31}+p_{34}q_{12} = 0. \tag{7}$$

[Let $\mathbf{y}, \mathbf{z}$ be points of $\mathbf{p}$, $\mathbf{u}, \mathbf{v}$ points of $\mathbf{q}$. $\mathbf{p}, \mathbf{q}$ intersect if and only if $\mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}$ are coplanar, *i.e.* $|y_1z_2u_3v_4| = 0$.

Expanding the determinant on the first two rows gives (7). *Or* express the fact that any two of the planes (2), and the corresponding ones for **q**, are concurrent, and use (4).]

3. If **q** is given dually as $\tilde{\boldsymbol{\omega}}$ (*i.e.* the $\varpi$'s are got by writing $q$ instead of $p$ in (6)), then (7) may be written $\Sigma_{r,s}\, p_{rs}\, \varpi_{rs} = 0$.

## 27. Parallelism

In **26** let $s$ be an ordinary line and let the coordinates of **y**, **z** be taken in the forms $(x', y', z', 1)$, $(\lambda, \mu, \nu, 0)$, so that **z** is the special point of $s$. Then **26** (3) gives

$$\left.\begin{aligned} p_{23} = \nu y' - \mu z', \quad p_{31} = \lambda z' - \nu x', \quad p_{12} = \mu x' - \lambda y', \\ p_{41} = \lambda, \quad p_{42} = \mu, \quad p_{43} = \nu, \end{aligned}\right\} \quad (1)$$

and reproduces the line-coordinates in **13**. Hence, *if* $(\lambda, \mu, \nu, 0)$ *is the special point of an ordinary line s, then s has direction* $(\lambda, \mu, \nu)$, *and conversely*. Accordingly *two ordinary lines are parallel if and only if they meet in a special point.*

Thence it follows: *An ordinary line is parallel to an ordinary plane if and only if it meets the plane in a special point; two ordinary planes are parallel if and only if they meet in a special line.*

We have now the following state of affairs: In $\bar{\mathscr{E}}$ the special points, lines, and plane satisfy the same incidence relations as ordinary points, lines, and planes; there are no exceptional elements. But, in recovering the geometry of $\mathscr{E}$, we recover the exceptional cases of parallelism by noticing any particular relations involving the special points and interpreting them in accordance with the results just given. It should be noted that these avoid the assignment of "direction" to a special line or the special plane.

**Sequence of points on a line.** In $\bar{\mathscr{E}}$ let $s, t$ be skew lines. Let a plane $\Pi$ through $t$ rotate continuously about $t$. In *every* position $\Pi$ meets $s$ in a unique point $P$ (**23** (i)), and every point of $s$ is a possible position of $P$. Hence in one complete rotation of $\Pi$, $P$ traverses con-

tinuously the whole line $s$ and returns to its starting-point. So, if $A$, $B$, $C$ are points of $s$, $P$ can start from $C$, traverse the positions $A$, $B$ once and only once, and return to $C$. Therefore we conclude that in $\overline{\mathcal{E}}$ the points $A$, $B$ divide $s$ into *two* segments $[AB]$ and $[AB]'$, say. Now let $s$ be ordinary and $A$, $B$ be ordinary points of $s$, and let $K$ be the special point of $s$. Then $K$ lies in one of the two segments, say $[AB]'$. When we revert to $\mathcal{E}$, we *cut out* $K$; when $P \equiv K$ in $\overline{\mathcal{E}}$, $\Pi$ is parallel to $s$ and does not meet $s$ in $\mathcal{E}$. So $[AB]$ becomes the unique segment $AB$ in $\mathcal{E}$, and provides the only route between $A$, $B$ along $s$.

**Points at infinity.** We now adopt more usual terms and call the special points, lines, and plane, *points, lines, and plane at infinity*. These are to have no connotation other than has been given in this chapter. We shall in future denote the plane at infinity by $\Omega$.

1. The planes $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ are parallel if and only if $\eta_1/\zeta_1 = \eta_2/\zeta_2 = \eta_3/\zeta_3$. Deduce that they are parallel if and only if $\Omega$ contains their common line, *i.e.* if they meet in a line at infinity.

## 28. Harmonic Ranges and Pencils

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}'$, $\mathbf{x}''$ be four collinear points. By **23** (2) there exist numbers $\lambda'$, $\mu'$, $\lambda''$, $\mu''$ such that

$$\mathbf{x}' = \lambda'\mathbf{u}+\mu'\mathbf{v}, \qquad \mathbf{x}'' = \lambda''\mathbf{u}+\mu''\mathbf{v}. \quad . \quad (1)$$

We call the quantity

$$(\mu' : \lambda')/(\mu'' : \lambda'') \equiv k \equiv (\mathbf{uv}, \mathbf{x}'\mathbf{x}'') \quad . \quad (2)$$

the *cross-ratio* of the "range" $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}'$, $\mathbf{x}''$. In particular, if $k = -1$ we call the range *harmonic* and call $\mathbf{x}'$, $\mathbf{x}''$ harmonic conjugates w.r.t. $\mathbf{u}$, $\mathbf{v}$.

Dual definitions apply to a "pencil" of four collinear planes.

Now let $s$, $t$ be skew lines, $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$ points of $s$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$

planes through $t$. Then there exist numbers $\lambda$, $\mu$, $l$, $m$, such that

$$\mathbf{x} = \lambda\mathbf{u}+\mu\mathbf{v}, \qquad \boldsymbol{\xi} = l\boldsymbol{\eta}+m\boldsymbol{\zeta}. \quad . \quad . \tag{3}$$

Further, let $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ contain $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$ respectively, so that the fundamental incidence relation **24** (1) gives

$$\boldsymbol{\xi}\,.\,\mathbf{x} = 0, \qquad \boldsymbol{\eta}\,.\,\mathbf{u} = 0, \qquad \boldsymbol{\zeta}\,.\,\mathbf{v} = 0. \quad . \tag{4}$$

Combining (3), (4) we obtain

$$(l\boldsymbol{\eta}+m\boldsymbol{\zeta})\,.\,(\lambda\mathbf{u}+\mu\mathbf{v}) = l\mu\boldsymbol{\eta}\,.\,\mathbf{v}+m\lambda\boldsymbol{\zeta}\,.\,\mathbf{u} = 0,$$

whence

$$(m : l)/(\mu : \lambda) = k, \quad . \tag{5}$$

where $k$ is independent of $\lambda$, $\mu$, $l$, $m$. Hence if $\lambda'$, $\mu'$, $l'$, $m'$, $\lambda''$, $\mu''$, $l''$, $m''$ are two sets of corresponding values of these parameters, (5) gives

$$(m' : l')/(m'' : l'') = (\mu' : \lambda')/(\mu'' : \lambda'').$$

Therefore *the cross-ratio of a pencil of planes is equal to that of the range of points in which the planes meet any transversal.*

1. The definition (2) gives the cross-ratio as ordinarily defined. [Use **23** 3.]

2. If $(\mathbf{uv}, \mathbf{xy}) = -1$, then

$$(\mathbf{xy}, \mathbf{uv}) = (\mathbf{uv}, \mathbf{yx}) = \text{etc.} = -1.$$

3. If $\mathbf{u}$, $\mathbf{v}$ are ordinary and $(\mathbf{uv}, \mathbf{xy}) = -1$, then when $\mathbf{y}$ is the point at infinity on $\mathbf{uv}$, $\mathbf{x}$ is the midpoint of the segment $\mathbf{uv}$, and conversely.

CHAPTER V

# GENERAL EQUATION OF THE SECOND DEGREE *

RESULTS in this chapter, unless otherwise stated, apply to $\bar{\mathscr{E}}$.

## 29. Lemmas on Conics

It will be assumed that the reader is familiar with the following results of *plane geometry*, here summarised for convenient reference:

Confine attention to any plane $\Pi$ in $\bar{\mathscr{E}}$, and let the points *in this plane* be labelled by any system of homogeneous point-coordinates. A *conic* in $\Pi$ will be defined as the locus $\Gamma$ of the equation got by equating to zero an indefinite quadratic form in these coordinates. (*Quadratic form* is defined in Aitken, p. 20; it is *indefinite* if it vanishes for any values of the variables not all zero.)

I. If the quadratic form is irreducible to linear factors, $\Gamma$ is a *non-singular*, or *proper*, conic. Any line in $\Pi$ meets $\Gamma$ in two distinct points (secant), *or* in one point (tangent), *or* in no point (non-secant); at every point of $\Gamma$ there is a unique tangent. $\Gamma$ is a unicursal curve which does not cross itself; it separates the points of $\Pi$, excluding the points on $\Gamma$, into two domains, the *interior* $D_i$ and the *exterior* $D_e$. Every line through a point of $D_i$ is a secant and separates $D_i$ into two unconnected portions. Through a point of $D_e$ there pass secants, non-secants, and two tangents.

II. If the quadratic form is reducible to two linear factors,

* A reader having difficulty with this chapter may proceed to Chapter VI and study the properties of quadrics deduced from their standard equations. In the light of this he should then try to satisfy himself as to the truth of the general results in Chapter V.

$\Gamma$ is a *singular*, or *degenerate*, conic, and three cases arise: (i) Factors real and distinct; $\Gamma$ is a *line-pair*. (ii) Factors real and coincident; $\Gamma$ is a *single line* (counted twice in the equation of $\Gamma$). (iii) Factors complex; $\Gamma$ is a *single point*.

III. In all cases it is convenient to define a tangent as a line which meets $\Gamma$ in one and only one point or which forms part of $\Gamma$. $P$ being a point of $\Gamma$, either a unique tangent goes through $P$, then called a *non-singular*, or *simple*, point, or else every line through $P$ is a tangent and $P$ is called a *singular*, or *double*, point. In I, $\Gamma$ has so singularity. In II (i) the meet of the two lines is the only singularity, II (ii) every point of $\Gamma$ is singular, II (iii) the single point of $\Gamma$ is singular.

IV. Suppose $\Gamma$ is proper. (i) Let $ABC$ be a self-polar triangle w.r.t. $\Gamma$; one vertex is in $D_i$, two are in $D_e$ and their join is a non-secant; every chord of $\Gamma$ through a vertex is divided harmonically by that vertex and the opposite side. (ii) Let $AB$ be the tangent at $B \equiv C$ on $\Gamma$; $C$ is the pole of $AB$ and there is no point such that every chord through it is divided harmonically by that point and $AB$; the polar of $A$ goes through $C$ and every chord through $A$ is divided harmonically by $A$ and this polar.

V. In IV let $AB$ be $\omega$, the line at infinity in $\Pi$, and let $\omega$ be omitted so that $\Pi$ becomes a plane in $\mathcal{E}$. Then IV (i) gives: Every chord through $C$ is bisected at $C$, the *centre* of $\Gamma$; every chord parallel to the (*conjugate*) diameters $CA$, $CB$ is bisected by $CB$, $CA$ respectively. If $C$ is in $D_i$, $\Gamma$ does not extend to infinity and remains a unicursal curve with a single interior domain; both conjugate diameters of any pair meet $\Gamma$ (*ellipse*). If $C$ is in $D_e$, $\Gamma$ extends to infinity and the tangents from $C$ become the asymptotes; $\Gamma$ is separated into two branches and its interior into two domains; of any pair of conjugate diameters one does, and one does not, meet $\Gamma$ (*hyperbola*). IV (ii) gives: $\Gamma$ has no centre; it extends to infinity but has no asymptote; it is a unicursal curve with one point omitted and with a single interior domain. Every chord parallel to a given direction is bisected by a line (*diameter*) parallel to the fixed direction determined by lines through $C$ (*parabola*).

1. Enumerate the possible types of degenerate conic in $\mathcal{E}$.

## 30. Quadric

The general linear form in $x_1, x_2, x_3, x_4$ is

$$\Pi \equiv \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 \equiv \sum_{r=1}^{4} \xi_r x_r, \quad . \qquad (1)$$

where $\xi_1, \ldots, \xi_4$ are any numbers not all zero. We have seen that the locus in $\overline{\mathscr{E}}$ of the equation $\Pi = 0$ is a plane, and that every plane has an equation of this form. We saw too that the plane $\Pi$ may be labelled by the set of numbers $\xi_r$, which serve as homogeneous coordinates for $\Pi$.

We shall now adopt the *summation convention* according to which we omit the summation symbol and understand that *whenever a suffix is repeated in any term that term is to be summed over every value of the repeated suffix* (in present cases over the values 1 to 4). Thus (1) is written as simply $\xi_r x_r$.

We shall also extend the device introduced in **8**, and speak of the locus $F$ of an algebraic form $F$, meaning thereby the locus of the equation got by equating that form to zero.

The general quadratic form in $x_1, x_2, x_3, x_4$ is

$$\left.\begin{aligned} S \equiv a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{14}x_1x_4 \\ + \ a_{22}x_2^2 \ \ + 2a_{23}x_2x_3 + 2a_{24}x_2x_4 \\ + \ a_{33}x_3^2 \ \ + 2a_{34}x_3x_4 \\ + \ a_{44}x_4^2, \end{aligned}\right\} . \qquad (2)$$

where $a_{11}, \ldots, a_{44}$ are any numbers (here assumed to be *real*) not all zero. According to the summation convention we write

$$S \equiv a_{rs}x_rx_s, \quad \text{where} \quad a_{rs} = a_{sr}, \quad . \qquad (3)$$

where summation over all $r$, $s$ is implied.* We call the locus $S$ (*i.e.*, as just explained, the locus of the equation

* The reader should at first transcribe such expressions in full in order to ensure his understanding the operation of the convention. In particular, he should convince himself that it is immaterial what letter is used for a repeated suffix, since changing

$S = 0$) in $\overline{\mathscr{E}}$, when it exists, a *quadric*. As in the case of the plane, we may label $S$ by the set of ten numbers $a_{rs}$, which would in fact serve as homogeneous coordinates of $S$. Now these numbers can be regarded as the elements of a symmetric square matrix $\mathsf{A} \equiv [a_{rs}]$, so the properties of the corresponding quadric must be expressible in terms of properties of this matrix. It is known, in fact, that the language of matrix theory is that most appropriate to express the properties of a quadratic form,* and the same must then apply to the corresponding quadric. For the properties of the quadric are simply those of the quadratic form when the latter are given a geometrical interpretation in accordance with the theory of Chapter IV.

1. There is at least one quadric through any *nine* given points; either it is unique or there exists an infinite number of such quadrics.

**Intersection of a quadric and a plane.** Let $\Pi$ be any plane and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ fixed non-collinear points of $\Pi$. Then (**23** (4)) every point $\mathbf{x}$ of $\Pi$ can be expressed in the form $\mathbf{x} = \lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w}$. This is also a point of $S$ if and only if

$$\Gamma \equiv a_{rs}(\lambda u_r+\mu v_r+\nu w_r)(\lambda u_s+\mu v_s+\nu w_s) = 0. \qquad (4)$$

Now $\Gamma$ is a quadratic form in $\lambda$, $\mu$, $\nu$. Also we have seen (**23** 9) that $\lambda$, $\mu$, $\nu$ serve as homogeneous coordinates *in the plane* $\Pi$. We know from plane geometry that the locus, when it exists, of a quadratic form in homogeneous coordinates in the plane is a *conic*. Hence, *if the plane* $\Pi$ *has any points in common with* $S$, *these points form in general a conic in* $\Pi$. A quadric is therefore in general a locus such that every plane section is a *curve*, namely a

the letter (in both places where it occurs) makes no difference when the expression is written in full. But the letter used for the repeated suffix must, of course, be different from any other suffix which may occur; *i.e.* the same literal suffix must not appear in more than two places, and it occurs in two places when, and only when, summation over all its numerical values is implied.

* *Cf.* Aitken, or almost any other book on modern algebra.

conic. It agrees with our intuitive notions to call such a locus a *surface*, thus giving a particular illustration from another standpoint of the conclusions of **11** 1.

We discover later the conditions which determine whether the conic (4) is proper or degenerate.

**Reducible quadric.** The preceding discussion may fail in one way, viz. the form $\Gamma$ may vanish identically. In this case *every* point of $\Pi$ satisfies (4) and is therefore a point of $S$. Hence if $\Pi$ has equation $\Pi \equiv \xi_r x_r = 0$, then $S = 0$ for every $\mathbf{x}$ for which $\Pi = 0$ and so $\Pi$ is a linear factor of $S$. Then the remaining factor must also be linear, say $\Lambda \equiv \eta_r x_r$.

Conversely, suppose $S$ does possess linear factors $\Pi$, $\Lambda$. Then $S$ consists entirely of the points which make $\Pi = 0$ or $\Lambda = 0$. Since the coefficients of $S$ are real, three cases arise:

(i) $\Pi$, $\Lambda$ are real and distinct; $S$ consists of the *plane-pair* $\Pi$, $\Lambda$.

(ii) $\Pi$, $\Lambda$ are real and coincident; $S$ consists of the *single plane* $\Pi$ (counted twice in the equation of $S$).

(iii) $\Pi$, $\Lambda$ are of the type $M \pm iN$, where $M$, $N$ are distinct real linear forms in $x_1$, $x_2$, $x_3$, $x_4$; then $\Pi = 0$ or $\Lambda = 0$ if and only if $M = 0$, $N = 0$ simultaneously; $S$ consists entirely of the *line* given by $M = 0$, $N = 0$.

In these cases $S$ is called a *reducible quadric.**

2. The general equation of a plane-pair involves *six* independent constants.

* The properties of reducible quadrics are trivial properties of lines and planes, and the reader may ask why we include them in what follows. It is to render the discussion sufficiently exhaustive for us to use the following type of argument: To prove that an irreducible quadric $S$ has property "$P$". We show first that it has either property "$P$" or property "$P'$". From some result established for reducible quadrics, we deduce that if $S$ has property "$P'$" it is reducible. The desired result is then established.

3. If $a_{rs}x_rx_s \equiv (\xi_rx_r)(\eta_sx_s)$ with $a_{rs} = a_{sr}$, then

$$2[a_{rs}] = \{\xi_r\}[\eta_s]+\{\eta_r\}[\xi_s],$$

where on the r.h.s. { } [ ] denote single-column and single-row matrices, respectively.

4. *If a plane $\Pi$ contains any set of points of $S$ which do not lie on a single conic, then $\Pi$ is entirely contained in $S$.*

**Intersection of a quadric and a line.** Let $s$ be any line and $\mathbf{y}$, $\mathbf{z}$ distinct fixed points of $s$. Then by **23** (2) any point $\mathbf{x}$ of $s$ can be expressed in the form

$$\mathbf{x} = \lambda\mathbf{y}+\mu\mathbf{z}. \qquad (5)$$

$\mathbf{x}$ is also a point of $S$ if and only if $a_{rs}x_rx_s = 0$, *i.e.*, from (5),

$$a_{rs}(\lambda y_r+\mu z_r)(\lambda y_s+\mu z_s) = 0$$

or, since $a_{rs} = a_{sr}$,

$$\lambda^2a_{rs}y_ry_s+2\lambda\mu a_{rs}y_rz_s+\mu^2a_{rs}z_rz_s = 0. \qquad (6)$$

This is called the *ratio-equation* (Joachimstahl's), since its roots in $\lambda : \mu$ yield the ratios (by **23** 3 actually $\mu z_4 : \lambda y_4$) in which the intersections of $s$, $S$ divide the segment $\mathbf{yz}$. Substituting these roots in (5) we obtain the actual points of intersection.

Since (6) is quadratic in $\lambda : \mu$ with real coefficients, we have the following possibilities:

(i) $a_{rs}y_ry_s$, $a_{rs}y_rz_s$, $a_{rs}z_rz_s$ not all zero; (6) has ($a$) no real root, or ($b$) two equal real roots, or ($c$) two distinct real roots;

(ii) $a_{rs}y_ry_s = a_{rs}y_rz_s = a_{rs}z_rz_s = 0$; (6) is satisfied by every arbitrary value of $\lambda : \mu$, and since (6) is of degree two, it is convenient to regard every such value as a double root of (6).

These yield for any line $s$ the four corresponding mutually exclusive possibilities:

(i) (a) $s$ does not meet $S$; $s$ is then called a *non-secant* of $S$;

(b) $s$ meets $S$ in precisely one point; $s$ is then a *tangent line* of $S$ at that point, which is its *point of contact* (see below);

(c) $s$ meets $S$ in precisely two distinct points; $s$ is then called a *secant* of $S$;

(ii) every point of $s$ is a point of $S$; $s$ then lies entirely in $S$ and is called a *generator* of $S$.

**Tangent line.** We shall now more exactly define a tangent line of $S$ as follows: *If the line $s$ meets the quadric $S$ in a point* $\mathbf{x}$ *given by* (5) *and if the corresponding value of* $\lambda : \mu$ *is a double root of the ratio-equation* (6), *then $s$ is a tangent line with point of contact* $\mathbf{x}$. It follows from the possibilities enumerated above that, if $s$ is a tangent line of $S$, then, either $s$ has a unique point of contact (case (i) (b)), or every point of $s$ is a point of contact and $s$ is a generator (case (ii)). Briefly, a tangent line is one which meets $S$ in only one point or is a generator of $S$.

Particular attention is invited to this definition. It facilitates the concise formulation of properties of quadrics, but it differs from the standard definition and is not immediately applicable to other surfaces. The following comments may be noted: The definition is preferable to one which speaks of a tangent line meeting the surface in "coincident" or "consecutive" points. In addition to lines which "touch" the surface in accordance with our intuitive notions, it leads us to regard, for instance, any line through the vertex of a quadric cone as a tangent line of the cone. The analogous definition of a tangent of a conic is given in **29** III.

5. Any tangent line $s$ at a point $\mathbf{y}$ of $S$ is also a tangent at $\mathbf{y}$ to the conic in which any plane through $s$ meets $S$, and conversely.

**Tangent plane.** Now let $\mathbf{y}$ be any fixed point of $S$, so that

$$a_{rs}y_r y_s = 0. \quad . \quad . \quad . \quad (7)$$

Let $\mathbf{z}$ be any fixed point of $\overline{\mathscr{E}}$ distinct from $\mathbf{y}$, and let $s$ be the join of $\mathbf{y}$, $\mathbf{z}$. Then any point of $s$ is again given by (5).

Since $s$ contains $\mathbf{y}$, and $\mathbf{y}$ is a point of $S$, then $s$ now certainly meets $S$ in $\mathbf{y}$. So $s$ is necessarily either a secant or else a tangent line having $\mathbf{y}$ as a point of contact. Since $\mathbf{x} \equiv \mathbf{y}$ in (5) requires $\mu = 0$, $s$ is therefore a tangent line if and only if $\mu = 0$ is a double root of (6). We seek now the geometrical condition which must be satisfied by $\mathbf{z}$ in order that $s$ shall in fact be a tangent line.

Using (7), the two roots of (6) are seen to be given by

$$\mu = 0, \qquad (8)$$

and

$$2\lambda a_{rs}y_r z_s + \mu a_{rs}z_r z_s = 0. \qquad (9)$$

((8) merely reproduces the fact that $s$ necessarily meets $S$ in $\mathbf{y}$.) So $\mu = 0$ is a double root, and $s$ is consequently a tangent line, if and only if (9) is satisfied by $\mu = 0$. This is true if and only if $a_{rs}y_r z_s = 0$ (since $\lambda$, $\mu$ cannot vanish together), *i.e.* $\mathbf{z}$ is any point $\mathbf{x}$ of the locus given by *

$$T \equiv a_{rs}y_r x_s = 0. \qquad (10)$$

$T$ is a linear form in $\mathbf{x}$, with coefficients $a_{rs}y_r$, and two cases arise:

(i) $a_{rs}y_r$ $(s = 1, \ldots, 4)$ are not all zero and we call $\mathbf{y}$ a *non-singular* or *simple* point of $S$. Then (10) is the equation of a plane $T$ through $\mathbf{y}$; it is called the *tangent plane* of $S$ at $\mathbf{y}$, and $\mathbf{y}$ is called the *point of contact* of $T$.

(ii) $a_{rs}y_r = 0$ $(s = 1, \ldots, 4)$ and we call $\mathbf{y}$ a *singular* or *double* point, or *singularity*, of $S$. Then (10) is satisfied by *every* point $\mathbf{x}$ of $\overline{\mathcal{E}}$.

We have therefore proved the following important theorems:

*Any point* $\mathbf{y}$ *of* $S$ *is either non-singular or singular.*

* Note that, since $a_{rs} = a_{sr}$, then $a_{rs}y_r x_s \equiv a_{rs}x_r y_s$, the sum on each side, implied by the repeated suffix, containing precisely the same terms, though in different orders. The reader should verify this by writing the expressions in full.

*If* $\mathbf{y}$ *is non-singular, there exists a tangent plane at* $\mathbf{y}$ *such that every tangent line at* $\mathbf{y}$ *lies in this plane and every line lying in this plane and passing through* $\mathbf{y}$ *is a tangent line at* $\mathbf{y}$.

*Conversely, if every tangent line at* $\mathbf{y}$ *lies in one plane,* $\mathbf{y}$ *is non-singular.*

*If* $\mathbf{y}$ *is singular, then every line through* $\mathbf{y}$ *is a tangent line at* $\mathbf{y}$.

*Conversely, if not every tangent line at* $\mathbf{y}$ *lies in one plane,* $\mathbf{y}$ *is singular.*

If $\mathbf{z}$ is a fixed point not on $S$, then from (10) the tangent plane at $\mathbf{y}$ contains $\mathbf{z}$ if and only if $a_{rs}y_r z_s = 0$, *i.e.* $\mathbf{y}$, besides being a simple point of $S$, is a point $\mathbf{x}$ of the plane

$$a_{rs}x_r z_s = 0. \qquad . \qquad . \qquad . \qquad (11)$$

Hence *the points of contact of tangent planes which pass through* $\mathbf{z}$ *are those non-singular points* (if any), *and only those, in which the plane* (11) *meets* $S$. This is called the *plane of contact* of $\mathbf{z}$.

**Singular and non-singular quadrics.** We have just seen that $\mathbf{y}$ is a singularity of $S$ if and only if it lies on $S$ and if it satisfies $a_{rs}y_r = 0$ $(s = 1, \ldots, 4)$, *i.e.* $\mathbf{y}$ is a common point of the four planes

$$a_{rs}x_r = 0. \quad (s = 1, \ldots, 4) \qquad (12)$$

Now a necessary and sufficient condition for $\mathbf{y}$ to exist satisfying (12) is

$$\Delta \equiv |\mathsf{A}| \equiv \qquad = 0, \qquad (13)$$

*i.e.* that the matrix $\mathsf{A}$ be singular (Aitken, p. 53). Moreover, if (13) is satisfied and $\mathbf{y}$ is any solution, then substituting $\mathbf{x} = \mathbf{y}$ in (12), multiplying (12) by $y_s$, and adding the four equations, we obtain $a_{rs}y_r y_s = 0$, *i.e.* $\mathbf{y}$ is necessarily a point of $S$.

More particularly, we assert: *

*If rank* $\mathsf{A} = 4$, the planes (12) have no common point; *$S$ has no singularity.*

*If rank* $\mathsf{A} = 3$, the planes (12) have a unique common point; *$S$ has a unique singularity.*

*If rank* $\mathsf{A} = 2$, the planes (12) have a line in common; *$S$ has a line of singularities.*

*If rank* $\mathsf{A} = 1$, the planes (12) are all identical; *$S$ has a plane of singularities.*

*If rank* $\mathsf{A} = 0$, every $a_{rs}$ is zero, which is *contrary to hypothesis.*

$S$ is called a *non-singular quadric* if it possesses no singularity; it is called a *singular quadric* if it possesses one or more singularities.

**Properties of singularities.** The algebraic proofs of (a), (b) below are trivial; that of the first part of (d) has just been indicated. Nevertheless, for the sake of uniformity and of gaining insight into geometrical results, we establish them by geometrical arguments (based, of course, on the earlier algebra). Analogous considerations govern the choice of method in other places also, and the reader should, wherever possible, supply the analytical proofs.

(a) *The join of a singularity* $\mathbf{y}$ *to any other point of $S$ is a generator.* For every line through $\mathbf{y}$ either meets $S$ in no other point or is a generator. It follows that, if $S$ is singular, it can be regarded as generated entirely by the aggregate of generators which pass through any singularity.

(b) *Every tangent plane contains every singularity of $S$.* For let $\mathbf{z}$ be any non-singular point of $S$ and $\mathbf{y}$ any singularity. By (a) $\mathbf{zy}$ is a generator through $\mathbf{z}$. Therefore the tangent plane at $\mathbf{z}$ contains $\mathbf{zy}$ and so contains $\mathbf{y}$.

(c) In the cases of *reducible quadrics* on p. 58 we have respectively, as immediate consequences of the geometrical character of a singularity: (i) *Every point on the meet of*

* *Cf.* Aitken, 26–28.

$\Pi$, $\Lambda$ *is singular; every other point of* $S$ *is non-singular.* (ii) and (iii) *Every point of* $S$ *is singular.*

(d) *A quadric* $S$ *has* (i) *no singularity*, or (ii) *a unique singularity*, or (iii) *a line of singularities*, or (iv) *a plane of singularities;* in (i), (ii) $S$ *is irreducible*, in (iii), (iv) $S$ *is reducible.* If $\Delta \neq 0$, we have (i). If $\Delta = 0$ we either have (ii) or $S$ possesses at least two singularities $\mathbf{y}$, $\mathbf{z}$. In the latter case $\mathbf{yz}$ is a generator, by (a). If $S$ contains no point not on this generator, then by (c) all its points are singular, and we have (iii). If $S$ contains a point $\mathbf{u}$ not on this generator, then by (*a*) $\mathbf{uy}$, $\mathbf{uz}$ are also generators; therefore the plane of $\mathbf{y}$, $\mathbf{z}$, $\mathbf{u}$ contains three generators and so by 4 it is part of $S$. If $S$ contains no point not in this plane, then by (c) all its points are singular, and we have (iv). If $S$ contains a point $\mathbf{v}$ not in this plane, then by a repetition of the argument, the plane of $\mathbf{y}$, $\mathbf{z}$, $\mathbf{v}$ is part of $S$; therefore $S$ is a plane-pair, and by (c) we again have (iii).

(e) *If* $S$ *possesses a unique singularity* $\mathbf{y}$, *then, either there is no other point of* $S$, *or* $S$ *is generated by the joins of* $\mathbf{y}$ *to the points of a proper conic in a plane not containing* $\mathbf{y}$. For, if $S$ contains any point other than $\mathbf{y}$, then by (a) it is entirely generated by lines through $\mathbf{y}$. Any plane $\Pi$ not containing $\mathbf{y}$ meets each of these lines in a single point; the aggregate of these points gives the intersection of $\Pi$, $S$ and therefore a conic. Had this conic any of the degenerate forms **29** II, it is easy to see that $S$ would be reducible and so would contain more than one singularity. Hence the conic is proper. In this case $S$ is called a *quadric cone*, and $\mathbf{y}$ its *vertex*.

Conversely, *if* $\mathbf{y}$ *is any point and* $\Gamma$ *any proper conic in a plane* $\Pi$ *not containing* $\mathbf{y}$, *then the joins of* $\mathbf{y}$ *to the points of* $\Gamma$ *generate a quadric cone with vertex* $\mathbf{y}$. For let $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(5)}$ be five points of $\Gamma$, and $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$ be points, not in $\Pi$, of three of the joins of $\mathbf{y}$ to points of $\Gamma$. By 1 there exists a quadric $S$ through the nine points $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(5)}, \mathbf{v}^{(1)}, .., \mathbf{v}^{(3)}, \mathbf{y}$. $S$ meets $\Pi$ in a conic containing five points of $\Gamma$ and so identical with $\Gamma$. $S$ then contains three points of each of

the joins of $\mathbf{y}$ to $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$, and therefore contains these joins. Hence $\mathbf{y}$ is a singularity of $S$ by 6. This singularity is unique; otherwise by (d) $S$ would be reducible and would meet $\Pi$ in a degenerate conic. Thus $S$ is a quadric cone with vertex $\mathbf{y}$. Since $\Gamma$ is a plane section, $S$ is generated by the joins of $\mathbf{y}$ to points of $\Gamma$.

6. If three generators go through $\mathbf{y}$, then $S$ is reducible or $\mathbf{y}$ is a singularity. [If they are coplanar, their plane is part of $S$; if they are not coplanar, $\mathbf{y}$ is a singularity.]

7. If $S$ possesses a singularity $\mathbf{y}$ and a generator $s$ not containing $\mathbf{y}$, then $S$ contains the plane of $\mathbf{y}$ and $s$.

8. If the tangent planes at four non-coplanar points of $S$ are concurrent in $\mathbf{z}$, then $\mathbf{z}$ is a singularity. [If $\mathbf{z}$ is not on $S$, only tangent planes at points in the plane (11) can go through $\mathbf{z}$. Therefore $\mathbf{z}$ is on $S$, and from **31** I we deduce that at least three non-coplanar generators go through $\mathbf{z}$.]

## 31. Properties of Tangent Plane

The importance of the following theorems earns them a separate section.

Let $S$ possess at least one simple point $\mathbf{y}$ and let $T$ be the tangent plane at $\mathbf{y}$.

I. *If $T$ meets $S$ in any point* $\mathbf{z}$ *distinct from* $\mathbf{y}$, *then* $\mathbf{yz}$ *is a generator*. For every line lying in $T$ and going through $\mathbf{y}$ either meets $S$ in no other point or is a generator of $S$.

II. *$T$ meets $S$ in* (i) $\mathbf{y}$ *only*, or (ii) *one line $s$ through* $\mathbf{y}$, or (iii) *two lines $s$, $t$ through* $\mathbf{y}$, or (iv) *$T$ is part of $S$*. For, from I, $T$ either meets $S$ in $\mathbf{y}$ only or in generators through $\mathbf{y}$; and if $T$ contains more than two generators, by 6, $T$ is part of $S$.

III. (i) *If $T$ meets $S$ in* $\mathbf{y}$ *only, then $S$ is non-singular and possesses no generators.* For, by p. 63 (b), $T$ contains every singularity. But $T$ meets $S$ in $\mathbf{y}$ only and $\mathbf{y}$ is non-singular. Therefore $S$ possesses no singularity. Further, had $S$ any generator, this would meet $T$ in a point of $S$ and so could do so only in $\mathbf{y}$. But, since $\mathbf{y}$ is non-singular, any generator

through $\mathbf{y}$ must lie in $T$, thus giving a contradiction. Therefore $S$ possesses no generator.

(ii) *If $T$ meets $S$ in a unique line $s$, then $S$ is a cone and $T$ is the tangent plane at every point of $s$ except the vertex.* Every line in $T$ is a tangent line; for it either meets $s$ and consequently $S$ in one and only one point or is the generator $s$. Therefore $T$ is the tangent plane at every non-singular point of $s$. Moreover, $s$ contains at most one singularity, and $S$ possesses non-singular points not on $s$; otherwise by pp. 63–4 (c), (d) every point of $s$, including $\mathbf{y}$, would be singular. Let then $\mathbf{u}$ be a non-singular point not on $s$. Then the tangent plane at $\mathbf{u}$ meets $s$ in some point $\mathbf{z}$. By (a), $\mathbf{zu}$ is a generator, and so not every tangent line through $\mathbf{z}$ lies in one plane. Therefore $\mathbf{z}$ is a singularity and is unique. Hence $S$ is by p. 64 (e) a quadric cone.

(iii) *If $T$ meets $S$ in precisely two lines $s$, $t$, then $S$ is non-singular and possesses two generators through every point of $S$.* For, were $\mathbf{z}$ any singularity, it would by p. 63 (b) lie in $T$, and by p. 63 (a) the join of every point of $s$, $t$ to $\mathbf{z}$ would be a generator. Since $s$, $t$ are the only generators in $T$, $\mathbf{z}$ could then only be the meet of $s$, $t$. But, by II (iii) this is $\mathbf{y}$ and is non-singular; so we should have a contradiction. Therefore a tangent plane exists at every point $\mathbf{u}$ of $S$, and meets $S$ either in $\mathbf{u}$ alone or in two generators through $\mathbf{u}$; otherwise the preceding results would show $S$ to be singular. But if it meets $S$ in $\mathbf{u}$ alone, then by III (i) $S$ possesses no generators at all, contrary to the postulated existence of $s$, $t$. Hence $S$ possesses two generators through every point $\mathbf{u}$. Further, by 6 not more than two generators go through any point, since $S$ is non-singular.

(iv) *If $T$ is part of $S$, then $S$ is a plane-pair.* For if $S$ contains one plane then it either consists entirely of that plane or is a plane-pair. But in the former case every point of $S$, including $\mathbf{y}$, would be singular.

IV. If a surface is entirely generated by lines it is said to be *ruled*. From the preceding work it follows:

*If any quadric possesses at least one generator, then it is a ruled quadric.*

1. The point of contact of a given tangent plane is unique if and only if the quadric is non-singular.

2. The discussion on p. 58 can now be completed as follows: (i) If $\Pi$ does not meet $S$, *or* meets $S$ in a proper conic, $\Pi$ is not a tangent plane and contains no singularity. (ii) If $\Pi$ meets $S$ in a unique point $\mathbf{y}$, either $\Pi$ is the tangent plane at $\mathbf{y}$ or $\mathbf{y}$ is singular. (iii) If $\Pi$ meets $S$ in a unique line $s$, either $\Pi$ is the tangent plane at every point of $S$ except one which is singular, or every point of $s$ is singular. (iv) If $\Pi$ meets $S$ in distinct lines $s$, $t$ with common point $\mathbf{y}$, either $\Pi$ is the tangent plane at $\mathbf{y}$ or $\mathbf{y}$ is singular.

3. Every plane through a generator $s$ of a non-singular quadric is a tangent plane at one and only one point $\mathbf{y}$ of $s$ and meets $S$ in a second generator through $\mathbf{y}$. [Also, of course, the tangent plane at every point of $s$ contains $s$.]

4. *An irreducible quadric $S$ is a rational algebraic surface, i.e.* the homogeneous coordinates of a variable point $\mathbf{x}$ of $S$ are expressible as polynomials in two parameters. [For, as in **19** for the case of a sphere, we can employ stereographic projection from a fixed point $\mathbf{y}$ of $S$ on to any fixed plane $\Pi$ to set up a (1–1) correspondence between the points of $S$ and those of $\Pi$. This fails only for points of $S$ in the tangent plane at $\mathbf{y}$.]

5. The plane of contact of any point contains every singularity of $S$.

## 32. Classification of Quadrics in $\bar{\mathscr{E}}$

**Existence.** The properties derived for non-singular quadrics are those they must exhibit *if* they exist. To establish the existence of such quadrics not possessing generators we have merely to note that the sphere is a particular example of this type.

Now let $g_1$, $g_2$, $g_3$ be three non-intersecting lines. Through nine points, three on each of $g_1$, $g_2$, $g_3$, there passes by **30** 1 at least one quadric $S$. Since each of $g_1$, $g_2$, $g_3$ meets $S$ in three points it lies entirely in $S$. Moreover $S$ is non-

singular, for by p. 64 (d) no type of singular quadric can possess three non-intersecting generators. Hence we have established the existence of non-singular ruled quadrics.

The existence of every type of singular quadric mentioned has been explicitly or implicitly established.

**Classification.** We can now extract from the work up to this stage the exhaustive classification of quadrics in $\overline{\mathscr{E}}$ given in Table 1. The rest of this chapter provides some elaboration of the properties in $\overline{\mathscr{E}}$ and the deduction of those in $\mathscr{E}$.

### 33. Non-singular Quadric

Throughout this section the quadric $S$ is assumed to be non-singular.

**Polar plane.** Let $\mathbf{y}$, $\mathbf{z}$ be such that the ratio-equation **30** (6) has real roots $\lambda' : \mu'$, $\lambda'' : \mu''$. Then $\mathbf{y}$, $\mathbf{z}$ are *harmonic conjugates* w.r.t. the points in which their join meets $S$ (and consequently $\mathbf{y}$, $\mathbf{z}$ are not themselves points of $S$) if, from **28**, $\lambda'/\mu' + \lambda''/\mu'' = 0$, *i.e.* if the sum of the roots of **30** (6) is zero, giving

$$a_{rs}y_r z_s = 0. \qquad . \qquad . \qquad . \qquad (1)$$

Hence, if $\mathbf{y}$ is fixed, $\mathbf{z}$ must be a point $\mathbf{x}$ of the plane

$$a_{rs}y_r x_s = 0. \qquad . \qquad . \qquad . \qquad (2)$$

Conversely, if $\mathbf{z}$ is any point of this plane such that $\mathbf{yz}$ meets $S$ in distinct points, $\mathbf{y}$, $\mathbf{z}$ are harmonic conjugates w.r.t. these points. We call (2) the *polar plane* of $\mathbf{y}$ w.r.t. $S$, and $\mathbf{y}$ the *pole* of the plane.

Note that the polar plane $\Pi$ of $\mathbf{y}$ is not completely defined as the locus of harmonic conjugates of $\mathbf{y}$ w.r.t. the pairs of points in which lines through $\mathbf{y}$ meet $S$. This locus is contained in $\Pi$, but is not in general the whole of $\Pi$, since in general not every join of $\mathbf{y}$ to a point of $\Pi$ meets $S$.

The derivation fails if $\mathbf{y}$ is on $S$. But, if $\mathbf{y}$ is on $S$, then (2) is the tangent plane at $\mathbf{y}$. We therefore *define* the

TABLE 1

| | | | | *Singularities.* | *Generators.* |
|---|---|---|---|---|---|
| Quadric $S$ | **Irreducible** | **Non-singular** | **Ruled** | None | Two through each point of $S$. |
| | | | **Not ruled** | None | None. |
| | | **Singular** | **Cone** | Unique singularity (vertex) | One through each point of $S$ other than vertex. All go through vertex. |
| | | | **Single point** | Unique singularity | None. |
| | **Reducible** | **Singular** | **Plane-pair** | Line of singularities (common line of planes) | All lines in each plane. |
| | | | **Single plane** (counted twice in equation of $S$) | Plane of singularities | All lines in plane. |
| | | | **Single line** | Line of singularities | $S$ consists of single generator. |

polar plane of $\mathbf{y}$ in that case as the tangent plane at $\mathbf{y}$. Since then the algebraic form is the same in all cases, the ensuing properties of polar planes hold without exceptions when the definition is extended in this manner.

We now have the theorems:

(i) *Every plane in $\overline{\mathscr{E}}$ has a unique pole w.r.t. $S$.* For a given plane $\xi_s x_s = 0$ is the polar plane (2) of $\mathbf{y}$ if

$$a_{rs} y_r = k\xi_s. \qquad (s = 1, \ldots, 4;\ k \neq 0) \qquad (3)$$

Since $S$ is non-singular, $|a_{rs}| \neq 0$, and the equations (3) have a unique solution $\mathbf{y}$. We recover as a corollary the result in **31 1** that every tangent plane of $S$ has a unique point of contact.

(ii) *The necessary and sufficient condition for a plane to be a tangent plane is that it should contain its pole.* For (2) contains $\mathbf{y}$ if and only if $a_{rs} y_r y_s = 0$, *i.e.* $\mathbf{y}$ is a point of $S$, and then (2) is the tangent plane at $\mathbf{y}$. It follows from **31** that the polar plane of a point not on $S$ either does not meet $S$ or meets it in a *proper* conic.

(iii) *The polar plane of* $\mathbf{y}$ *is the plane of contact of* $\mathbf{y}$, as follows from **30** (11).

(iv) *If the polar plane of* $\mathbf{y}$ *contains* $\mathbf{z}$, *then the polar plane of* $\mathbf{z}$ *contains* $\mathbf{y}$, since $a_{rs} y_r z_s \equiv a_{rs} z_r y_s$. We call $\mathbf{y}$, $\mathbf{z}$ *conjugate points*, and their polar planes *conjugate planes*, w.r.t. $S$.

(v) **Polar lines.** *The polar plane of every point of a fixed line s passes through another fixed line t, and the polar plane of every point of t passes through s.* Let $\mathbf{y}$, $\mathbf{z}$ be any distinct points of $s$. By (i) their polar planes are distinct, and so meet in a line $t$. By (iv) the polar plane of every point of $t$ contains $\mathbf{y}$, $\mathbf{z}$, and so contains $s$. Similarly the polar plane of every point of $s$ contains $t$. Each of $s$, $t$ is called the *polar line* of the other.

(vi) **Self-polar tetrahedron.** Take any point $\mathbf{y}$ not on $S$ and let $\Pi$ be its polar plane. Take any point $\mathbf{z}$ on $\Pi$ and let $s$ be the line in which the polar plane of $\mathbf{z}$ meets $\Pi$. Take any point $\mathbf{u}$ on $s$ and let $\mathbf{v}$ be the point in which the polar plane of $\mathbf{u}$ meets $s$. It follows from

(iv), (v) that, in the tetrahedron **yzuv**, *each vertex is the pole of the opposite face, and each edge is the polar line of the opposite edge;* it is called *self-polar.*

1. If any plane $\Pi$ through **y** meets $S$ in a conic $\Gamma$, then $\Pi$ meets the polar plane of **y** in the polar of **y** w.r.t. $\Gamma$ (with the definition of polar used in plane geometry). If **y**, **z** in $\Pi$ are conjugate w.r.t. $S$, they are conjugate w.r.t. $\Gamma$, and conversely.

2. If any plane $\Pi$ through $s$ meets $S$ in a conic $\Gamma$, then $\Pi$ meets the polar line of $s$ in the pole of $s$ w.r.t. $\Gamma$.

3. If any plane through **y**, **z** in (vi) meets $S$ in a conic $\Gamma$ and meets **uv** in **w**, then **yzw** is a self-polar triangle w.r.t. $\Gamma$.

4. If a proper conic $\Gamma$ is a section of $S$ and **yzw** is any self-polar triangle w.r.t. $\Gamma$, then there exists a self-polar tetrahedron w.r.t. $S$ having **y**, **z** as vertices, any plane through **yz** as one face, and the edge opposite **yz** containing **w**.

5. If a line $s$ contains the pole of a plane $\Pi$, then $\Pi$ contains the polar line of $s$.

6. *$s$ is a tangent line of $S$ if and only if it meets its polar line $t$,* and then $t$ is also a tangent line. Then, if $s \not\equiv t$, the plane of $s$, $t$ is the tangent plane of $S$ at the meet of $s$, $t$; $s$ is a generator if and only if $s \equiv t$.

7. The polar planes of **y**, **z** have plane-coordinates $a_{rs}y_s$, $a_{rs}z_s$ ($r = 1, \ldots, 4$). The meet of these planes has therefore from **26** (5) line-coordinates of the second kind $\varpi_{ij}$ where

$$\varpi_{ij} = a_{is}y_s a_{jt}z_t - a_{jt}y_t a_{is}z_s = a_{is}a_{jt}(y_s z_t - y_t z_s) = a_{is}a_{jt}p_{st},$$

where, from **26** (3), $p_{st}$ are the line-coordinates of the first kind of **yz**.

8. Combining 6, 7 with **26** 3, *the necessary and sufficient condition for the line* **p** *having polar line* $\tilde{\boldsymbol{\omega}}$ *to be a tangent line of $S$ is $p_{ij}\varpi_{ij} = 0$, i.e.*

$$a_{is}a_{jt}p_{ij}p_{st} = 0 \quad . \quad . \quad . \quad (4)$$

(summation over all values of $i, j, s, t$ from 1 to 4 being implied). (4) is called the *line equation* of the quadric $S$.

9. Let **p**, **q** be two given lines; then the polar line of **p** meets **q** if and only if

$$a_{is}a_{jt}p_{ij}q_{st} = 0.$$

Deduce that if the polar line of **p** meets **q**, then the polar line of **q** meets **p**. Then **p**, **q** are called *conjugate lines* w.r.t. $S$.

**Tangent cone.** We saw that the line $s$ involved in the ratio-equation **30** (6) is a tangent line of $S$ if and only if that equation has equal roots. This condition gives $(a_{rs}y_r z_s)^2-(a_{rs}y_r y_s)(a_{rs}z_r z_s) = 0$, *i.e.*, if **y** is fixed, **z** must be a point **x** of the locus

$$(a_{rs}y_r x_s)^2-(a_{rs}y_r y_s)(a_{rs}x_r x_s) = 0, \tag{5}$$

if it exists. This locus is therefore the aggregate of tangent lines of $S$ which pass through **y**. Hence, if **y** is not on $S$, it consists of the joins of **y** to the points (if any) in which $S$ is met by the plane of contact of **y**; from p. 70 (ii), (iii) it follows that (5) is then a quadric cone, called the *tangent*, or *enveloping, cone* from **y** to $S$.

10. If **y** is on $S$, the second term in (5) is zero, and it reduces to the equation of the tangent plane at **y** (counted twice).

11. The summation convention allows us to write

$$(a_{rs}y_r x_s)^2 \equiv a_{rs}y_r x_s a_{uv}y_u x_v \equiv a_{ur}a_{vs}y_u y_v x_r x_s;$$
$$(a_{rs}y_r y_s)(a_{rs}x_r x_s) \equiv a_{uv}a_{rs}y_u y_v x_r x_s.$$

Hence (5) may be written $(a_{ur}a_{vs}-a_{uv}a_{rs})y_u y_v x_r x_s = 0$. Verify algebraically that **y** is a singularity of the surface given by this equation.

**Ruled quadric.** Now let $S$ be a non-singular ruled quadric.

(i) *Through every point of $S$ there pass two and only two generators; all such generators form two systems, every member of one meeting every member of the other and no member of the same system.* The first part has been given in **31**. Let $g$, $h$ be the generators through a point **y** of $S$; let **z** be any point of $S$ not on $g$, $h$, and $T$ the tangent plane at **z**. By **31** II, $T$ does not contain $g$ or $h$; therefore $T$ meets $g$, $h$ in single points **u**, **v**, respectively, distinct from **y**. Hence, by **31** I, **zu** is one of the generators, say $h'$, and **zv** the other, say $g'$, through **z**. Also $h'$ does not meet $h$, $g'$ does not meet $g$; otherwise $T$ would meet $g$, $h$ in further

points. Hence, of the two generators through any point of $S$, one is like $g$ and meets $h$, the other is like $h$ and meets $g$ (fig. 4). Call all those of the first system $[g]$, or $g$-generators, those of the second $[h]$, or $h$-generators. Just as $g$, $g'$ meet

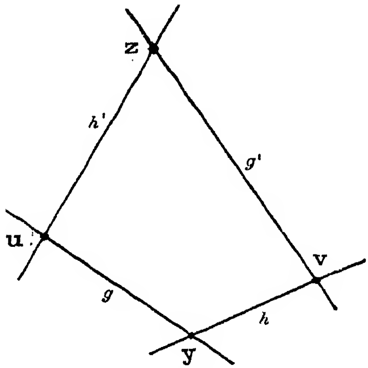


FIG. 4.

$h$, $h'$, every $g$-generator meets every $h$-generator; just as $g$, $g'$ do not intersect, no $g$-generator meets any other $g$-generator; similarly no $h$-generator meets any other $h$-generator.

Since every point of $S$ is on one and only one $g$-generator, $S$ is completely and non-redundantly generated by $[g]$. Similarly it is so generated by $[h]$. Each system is called a *regulus* of lines.

12. In fig. 4, **yz**, **uv** are polar lines w.r.t. $S$.

13. $g$, $g'$, $g''$ being any three $g$-generators, $[h]$ is completely determined as the aggregate of lines meeting $g$, $g'$, $g''$. [$g$, $g'$, $g''$ are skew lines; so through any point **y** of $g$ there passes a unique line $s$ meeting $g'$, $g''$. But through **y** there passes a unique generator $h$ meeting $g'$, $g''$. Hence $h \equiv s$. Further, we saw in **32** that, if $g_1$, $g_2$, $g_3$ are *any* three skew lines, there is at least one quadric $S$ having these for generators, and $S$ is non-singular. It now follows that *$S$ is the unique locus generated by a variable line meeting $g_1$, $g_2$, $g_3$*.]

(ii) *$S$ consists of one sheet; it is not divided into two portions by any plane.* For $\mathbf{y}$, $\mathbf{z}$ being any two points of $S$ as in fig. 4, the route $\mathbf{y} \to \mathbf{u} \to \mathbf{z}$, for example, along $g$, $h'$ lies wholly in $S$. Hence $S$ consists of a single connected portion, or *sheet*. Having regard to the nature of the sequence of points on a line (**27**), we can show moreover that, $\Pi$ being any specified plane not containing $\mathbf{y}$, $\mathbf{z}$, it is always possible to select a route from $\mathbf{y}$ to $\mathbf{z}$ which lies in $S$ and does not cross $\Pi$. Hence, in particular, $S$ still consists of a single sheet if the points common to $S$, $\Omega$ are omitted.

(iii) *Every plane* $\Pi$ (not being a tangent plane) *meets $S$ in a proper conic.* For $\Pi$ meets every line in $\overline{\mathscr{E}}$ and hence meets every generator of $S$. Therefore $\Pi$ meets $S$ in a conic, and this is not degenerate since otherwise, by **31** 2, $\Pi$ would be a tangent plane.

(iv) *Through every point* $\mathbf{w}$ *of $\overline{\mathscr{E}}$ not on $S$ there pass secants, tangent lines, and non-secants of $S$.* Let $g$ be any generator; the plane of $\mathbf{w}$, $g$ is the tangent plane at some point $\mathbf{y}$ of $g$; then $\mathbf{yw}$ is a tangent line at $\mathbf{y}$. Let $h$ be the other generator through $\mathbf{y}$, and $\Pi$ any plane through $\mathbf{yw}$ not containing $g$ or $h$. Then $\Pi$ is not a tangent plane and so, by (iii), meets $S$ in a proper conic $\Gamma$. By **30** 5, $\mathbf{yw}$ is a tangent of $\Gamma$. Hence, by **29** I, $\mathbf{w}$ is an exterior point of $\Gamma$, and so through $\mathbf{w}$ there pass secants, two tangents, and non-secants of $\Gamma$ in $\Pi$. These are also respectively secants, tangent lines, and non-secants of $S$.

14. Every point of $\overline{\mathscr{E}}$ is the vertex of a tangent cone of $S$.

15. *There is no interior domain of $S$ in $\overline{\mathscr{E}}$.* [There is no point such that every line through it is a secant of $S$.]

(v) *Through every secant of $S$ there pass two tangent planes; through a non-secant there passes no tangent plane.* A tangent plane meets $S$ in two lines; every other line in the plane meets each of these and so necessarily meets $S$. Hence, if a line $s$ is a non-secant, no tangent plane can

contain $s$. Now let $s$ be a secant meeting $S$ in $\mathbf{y}$, $\mathbf{z}$, and let $g$, $h$, $g'$, $h'$ be the generators through $\mathbf{y}$, $\mathbf{z}$. Once again we get a figure like fig. 4, and the tangent planes at $\mathbf{u}$, $\mathbf{v}$ contain $s$. No other tangent plane contains $s$, for such a plane would meet $S$ in further generators through $\mathbf{y}$, $\mathbf{z}$, which is impossible.

16. In (v) $g$, $g'$ are distinct, and $h$, $h'$ are distinct.

17. The polar line of a secant is a secant, that of a non-secant is a non-secant.

(vi) *In any self-polar tetrahedron* $\mathbf{yzuv}$, *one pair of opposite edges are non-secants, and the remaining edges are secants.* The plane $\mathbf{yzu}$ meets $S$ in a conic $\Gamma$ and the triangle $\mathbf{yzu}$ is self-polar w.r.t. $\Gamma$ (**33** 3). Hence by **29** IV one side, say $\mathbf{yz}$, is a non-secant of $\Gamma$ and so of $S$; the remaining sides are secants of $\Gamma$ and so of $S$. But $\mathbf{uv}$ is the polar line of $\mathbf{yz}$, and so by 17 is a non-secant; $\mathbf{yv}$, $\mathbf{zv}$ are the polar lines of $\mathbf{zu}$, $\mathbf{yu}$, and so by 17 are secants.

**Non-singular quadric without generators.** Now let $S$ be not ruled.

(i) *A plane is a tangent plane of $S$ if and only if it meets $S$ in a unique point. Any plane $\Pi$ containing more than one point of $S$ meets $S$ in a proper conic $\Gamma$.* (*Cf.* **31**.)

(ii) *$S$ separates the points of $\overline{\mathcal{E}}$* (excluding points of $S$) *into two domains, the interior $\Delta_i$, and the exterior $\Delta_e$; every line through a point of $\Delta_i$ is a secant of $S$; through any point of $\Delta_e$ there pass secants, tangent lines, and non-secants of $S$.* Let $\Pi$, $\Gamma$ be as in (i) and $D_i$, $D_e$ as in **29** I; let $\mathbf{u}$ be any point of $D_i$. Every line in $\Pi$ through $\mathbf{u}$ is a secant of $\Gamma$ and so of $S$. Hence every plane through $\mathbf{u}$, since it meets $\Pi$ in a line through $\mathbf{u}$, contains at least two points of $S$ and so meets $S$ in a proper conic. Hence no tangent plane, and so no tangent line, of $S$ goes through $\mathbf{u}$. Hence by **30** 5 no tangent of any section of $S$ goes through $\mathbf{u}$. Therefore $\mathbf{u}$ is an interior point of every section of $S$ by a plane through $\mathbf{u}$, and so *every* line through $\mathbf{u}$ is a secant of $S$.

Now let $\mathbf{u}'$ be another such point. Choose $\Pi$ to contain $\mathbf{u}, \mathbf{u}'$. Then $\mathbf{u}, \mathbf{u}'$ belong to a single domain, namely $D_i$, in $\Pi$. Therefore they belong to a single domain in $\overline{\mathscr{E}}$; this is $\Delta_i$.

Let $\mathbf{v}$ be a point of $D_e$. Through $\mathbf{v}$ can be drawn secants, tangents, and non-secants of $\Gamma$, and these are, respectively, secants, tangent lines, and non-secants of $S$. As before, we can show that all points like $\mathbf{v}$ belong to a single domain in $\overline{\mathscr{E}}$; this is $\Delta_e$.

18. Every point of $\Delta_e$, and no point of $\Delta_i$, is the vertex of a tangent cone of $S$.

19. From 18, the polar plane of a point of $\Delta_i$ does not meet $S$; the polar plane of a point of $\Delta_e$ meets $S$ in a proper conic, and conversely. [Note that this establishes the existence of planes which do not meet $S$.]

20. The polar line of a secant is a non-secant, and conversely.

21. Through any non-secant there pass two tangent planes [those at the points where its polar line meets $S$]; through any secant there passes no tangent plane.

(iii) *In any self-polar tetrahedron* $\mathbf{yzuv}$, *one vertex is in* $\Delta_i$, *three are in* $\Delta_e$; *all the faces through the former meet* $S$, *the face containing the latter does not meet* $S$. At least one vertex, say $\mathbf{y}$, is in $\Delta_i$; for if $\mathbf{z}, \mathbf{u}, \mathbf{v}$ are not, then by 19 the pole $\mathbf{y}$ of their plane is in $\Delta_i$. But if $\mathbf{y}$ is in $\Delta_i$ its polar plane does not meet $S$; so $\mathbf{z}, \mathbf{u}, \mathbf{v}$ are in $\Delta_e$.

(iv) $S$ *consists of a single sheet and is a closed surface; it is separated into two portions by any plane* $\Pi$ *which meets it in more than one point.* The first part follows directly from the existence of $\Delta_i$. The proof of the second part cannot be given here in full, but may be outlined as follows: Take any non-secant $s$ in $\Pi$; by 21, two tangent planes $T_1, T_2$ go through $s$. Then $S$ is entirely enclosed in one angle between $T_1$ and $T_2$. The surface $S$ is divided into a portion $S_{(1)}$ lying in one angle between $\Pi$ and $T_1$, and a portion $S_{(2)}$ lying in one angle between $\Pi$ and $T_2$.

In $\overline{\mathscr{E}}$ it is impossible to join a point within the first angle to a point within the second by a path which does not cross any of the planes. Therefore it is impossible to join a point of $S_{(1)}$ to a point of $S_{(2)}$ by a path which neither leaves $S$ nor crosses $\Pi$. In other words, when the points of $\Pi$ are omitted, $S_{(1)}$, $S_{(2)}$ are disconnected portions of $S$.

We now see the reason for the difference between this result and p. 74 (ii); a ruled quadric crosses any tangent plane and cannot be enclosed in an angle between two tangent planes.

## 34. Properties relative to the Plane at Infinity

**Properties of non-singular quadrics in general.** Let $S$ be a given non-singular quadric, $\Omega$ the plane at infinity, $C$ its pole w.r.t. $S$. We distinguish two cases.

(i) *$C$ not on $S$—Central quadric.* Every chord through $C$ is divided harmonically by $C$ and the point where it meets $\Omega$, and is therefore *bisected* at $C$. Also $C$ is the only point having this property. We now call $S$ a *central quadric*, $C$ the *centre*, a line through $C$ a *diameter*, a plane through $C$ a *diametral plane*.

Now let $CLMN$ be any self-polar tetrahedron having one vertex at $C$ and consequently the other vertices in $\Omega$. Any chord through $L$ is divided harmonically by $L$ and the point where it meets $CMN$, the polar plane of $L$. Therefore, using **27**, *every chord parallel to the diameter $CL$ is bisected by the diametral plane $CMN$*; similarly for the diameters $CM$, $CN$. We call $CL$, $CM$, $CN$ a triad of *conjugate diameters*, and we say that each is conjugate to the plane of the other two (or any parallel plane).

1. Let any plane $\Pi$ parallel to $CMN$, which meets $S$ in a conic $\Gamma$, meet $CL$ in $C'$, and $CLM$, $CLN$ in $C'M'$, $C'N'$. Then $C'$ is the centre, and $C'M'$, $C'N'$ are conjugate diameters, of $\Gamma$. [From what has just been proved, each of $C'M'$, $C'N'$ bisects all chords of $\Gamma$ parallel to the other.]

If now $C$, $L$ be fixed, $M$, $N$ may be *any* pair of conjugate points in the polar line of $CL$. Hence, using 1, all sections of $S$ parallel to the plane $CMN$ possess an infinite number of pairs of conjugate diameters such that, to any pair in one section there is a parallel pair in each of the others. By a theorem on conics it follows that all such sections are *similar and similarly situated conics.*

(ii) *$C$ on $S$—Non-central quadric.* Here $S$ touches $\Omega$ at $C$. There is now no point which is the midpoint of every chord through it. $S$ has no centre and is called *non-central.* But it is still convenient to call any line (not in $\Omega$) through $C$ a *diameter*, and any plane (other than $\Omega$) through $C$ a *diametral plane.* $C$ being a fixed point at infinity, all diameters are now parallel to a fixed direction.

Further, there is now no self-polar tetrahedron having one vertex at $C$. But let $M$, $N$ be any pair of conjugate points in $\Omega$, but not on $S$. Then the polar line of $MN$ goes through $C$ and meets $S$ again in $V$, say, so that the tangent plane at $V$ contains $MN$. We can now prove as in (i) that *every chord of $S$ parallel to $VM$ is bisected by the diametral plane $CVN$*, and every chord parallel to $VN$ is bisected by $CVM$. Thence we can deduce that all parallel sections of $S$ are similar and similarly situated conics with centres on a diameter of $S$.

**Ruled central quadric.** Let $S$ be ruled; comparing (i) with p. 75 (vi), we see that of any triad of conjugate diameters $CL$, $CM$, $CN$ meeting $\Omega$ in $L$, $M$, $N$, one is a non-secant; if this is $CL$, then $CM$, $CN$ are secants, $MN$ a non-secant, $LM$, $LN$ secants. Consequently, by p. 74 (iii), (v), every plane through $MN$ meets $S$ in a proper conic having no point at infinity, for were there any such point it would be a point of $S$ on $MN$; every plane through $LM$ or $LN$ (save the tangent planes through these lines) meets $S$ in a proper conic having two points at infinity, being the points of $S$ on $LM$ or $LN$. Combining these conclusions with **29** and (i), we have: *Given any triad of*

*conjugate diametral planes, all sections parallel to one of the planes are similar and similarly situated ellipses, all sections parallel to either of the other two are similar and similarly situated hyperbolas.* $S$ is called a *hyperboloid of one sheet.* There is a cone of contact with vertex $C$ and plane of contact $\Omega$, called the *asymptotic cone.*

2. A proper section of $S$ parallel to a tangent plane $T$ is a hyperbola whose asymptotes are parallel to the generators through the point of contact of $T$. [This fact provides a means of calculating the angle between the generators.]

**Ruled non-central quadric.** $\Omega$ is now a tangent plane meeting $S$ in two generators through $C$. Every plane through $C$, other than $\Omega$, meets $S$ in a proper conic having one point at infinity, *i.e.* a *parabola.* Every other plane, not a tangent plane, meets $S$ in a proper conic having two points at infinity (its intersections with the generators in $\Omega$), *i.e.* a *hyperbola.* A parallel tangent plane meets $S$ in a line-pair parallel to the asymptotes of the hyperbola. $S$ is called a *hyperbolic paraboloid.*

3. All the generators (except that in $\Omega$) of either system on $S$ are parallel to a fixed plane. [For they all meet one of the generators in $\Omega$, and so are parallel to any plane through it.]

**Non-ruled central quadric.** Let $S$ possess no generator; we have two cases:

(i) $\Omega$ *does not meet* $S$. $C$ is an interior point and every plane through $C$ meets $S$ in a proper conic having no point at infinity, *i.e.* an *ellipse.* $S$ is called an *ellipsoid.*

(ii) $\Omega$ *meets* $S$ (in a proper conic). $C$ is now an exterior point. Comparing p. 77 (i) with p. 76 (iii), we see that of any triad of conjugate diameters $CL$, $CM$, $CN$ meeting $\Omega$ in $L$, $M$, $N$, one is a secant; if this is $CL$, then $CM$, $CN$ are non-secants, $MN$ a non-secant, $LM$, $LN$ secants. Hence the diametral plane $CMN$ does not meet $S$, and there are two tangent planes through $MN$; every other plane through $MN$, which meets $S$, does so in a proper

conic having no point at infinity; every plane through $LM$ or $LN$ meets $S$ in a proper conic having two points at infinity. Interpreting these conclusions w.r.t. $\mathscr{E}$ by taking $\Pi$ in p. 76 (iv) to be $\Omega$ and then omitting its points, we have: *Given any triad of conjugate diametral planes, one does not meet $S$; there are two tangent planes parallel to this, and $S$ consists of two sheets having no part between these tangent planes; all sections parallel to this diametral plane are similar and similarly situated ellipses; all sections parallel to either of the other two are similar and similarly situated hyperbolas.* $S$ is called a *hyperboloid of two sheets.* Again there is an *asymptotic cone,* vertex $C$.

**Non-ruled non-central quadric.** $\Omega$ is now a tangent plane meeting $S$ in $C$ and nowhere else. Every plane through $C$ other than $\Omega$ meets $S$ in a proper conic having one point at infinity, *i.e.* a *parabola.* Every other plane section has no point at infinity, and so is an *ellipse.* $S$ is called an *elliptic paraboloid.*

## 35. Quadric Cone

For comparison with the exhaustive algebraic classification in Chapter VI, and for the sake of seeing what the results of **33, 34** become in degenerate cases, we give a brief account of singular quadrics in **35, 36.**

Let $S$ be a quadric cone with vertex $\mathbf{y}$. From **31,** *a plane is a tangent plane of $S$ if and only if it meets $S$ in a single line; any plane $\Pi$, not containing* $\mathbf{y}$, *meets $S$ in a proper conic* $\Gamma$.

*$S$ separates the points of* $\overline{\mathscr{E}}$ (excluding points of $S$) *into two domains, the interior $\Delta_i$, and the exterior $\Delta_e$; every line through a point of $\Delta_i$, not containing* $\mathbf{y}$, *is a secant; through any point of $\Delta_i$ there pass secants, tangent lines, and non-secants.* For let $\Pi$, $\Gamma$ be as stated above, and then let $D_i$, $D_e$ be as in **29** I; let $\mathbf{u}$ be any point of $D_i$. Every line in $\Pi$ through $\mathbf{u}$ is a secant of $\Gamma$ and so of $S$. Hence every plane through $\mathbf{u}$, since it meets $\Pi$ in a line through $\mathbf{u}$, contains a secant of $S$. Therefore no tangent plane, and so no tangent line, of $S$ goes through $\mathbf{u}$. Hence $\mathbf{u}$ is an interior point of every proper section of $S$ by a plane through $\mathbf{u}$, and the rest of the argument is as p. 75 (ii).

**Pole and polar.** The polar plane of a point $\mathbf{z}$ is defined in the same way as for a non-singular quadric, and so is given by

$$a_{rs}z_r x_s = 0. \quad . \quad . \quad . \quad (1)$$

This gives a unique plane unless $a_{rs}z_r = 0$ $(s = 1, \ldots, 4)$, *i.e.* unless $\mathbf{z}$ is the singularity $\mathbf{y}$. Let $\mathbf{u} = \lambda\mathbf{y} + \mu\mathbf{z}$ be any point on the join of $\mathbf{y}$, $\mathbf{z}$. Then, since $a_{rs}y_r = 0$ $(s = 1, \ldots, 4)$, the polar plane of $\mathbf{u}$ is

$$a_{rs}u_r x_r \equiv a_{rs}(\lambda y_r + \mu z_r)x_s \equiv \mu a_{rs}z_r x_s = 0,$$

which is the same as the polar plane of $\mathbf{z}$, $(\mu \neq 0)$. Further, since $a_{rs}y_s = 0$ $(r = 1, \ldots, 4)$, (1) is satisfied by $\mathbf{x} = \mathbf{y}$ for all $\mathbf{z}$.

These results mean that p. 70 (i) has to be replaced by: *Any plane* $\Pi$ *through* $\mathbf{y}$ *has a unique line of poles* $p$ *through* $\mathbf{y}$. We may call $p$ the *polar line* of $\Pi$, and $\Pi$ the polar plane of $p$. Any plane not containing $\mathbf{y}$ has no pole in the usual sense, but consistent results follow by defining the pole as the vertex (*cf.* **35** 2, **37** 3). Results (ii), (iii), (iv), p. 70, retain their validity, but (v) is replaced by: *The polar plane of every point of a fixed line* $s$*, not containing* $\mathbf{y}$*, passes through the polar line* $p$ *of the plane* $\Pi$ *containing* $s$ *and* $\mathbf{y}$. Since the polar plane of every point of $p$ is $\Pi$, we cannot define pairs of polar lines as before.

These results preclude the construction of a self-polar tetrahedron w.r.t. $S$. But take any line $p$, not a generator, through $\mathbf{y}$, and let $\Pi$ be its polar plane. Take any line $r$ in $\Pi$, not a generator, through $\mathbf{y}$, and let $\Lambda$ be its polar plane. $\Lambda$ contains $p$ and meets $\Pi$ in a line $s$ through $\mathbf{y}$; let $\Sigma$ be its polar plane. Then $\Sigma$ contains $p$, $r$. We call the resulting figure a *self-polar trihedron* w.r.t. $S$.

1. $S \equiv a_{rs}x_r x_s = 0$ is a cone if and only if $\mathsf{A} \equiv [a_{rs}]$ has rank 3. [Necessary and sufficient condition for a unique solution of $a_{rs}y_r = 0$ $(s = 1, \ldots, 4)$ (Aitken, **28**).]

2. Denoting by $A_{rs}$ the cofactor of $a_{rs}$ in $\Delta \equiv | a_{rs} |$, *the coordinates of the vertex* (the solution of $a_{rs}y_r = 0$ $(s = 1, \ldots, 4)$) *are* $A_{r1}$, $A_{r2}$, $A_{r3}$, $A_{r4}$, where $r$ can be any one of the numbers $1, \ldots, 4$, provided $A_{r1}$, $A_{r2}$, $A_{r3}$, $A_{r4}$ are not all zero. [From 1, not every $A_{rs}$ is zero; since $\Delta = 0$ the cofactors of any row are proportional to those of any other row.]

3. If $\Delta = 0$, $A_{44} = 0$, then $A_{r4} = 0$ $(r = 1, .., 4)$. Hence, from 2, the vertex is in $\Omega$ if and only if $A_{44} = 0$. [By Jacobi's theorem (Aitken, **42**), $A_{44}A_{33} - A_{34}^2 = (a_{11}a_{22} - a_{12}^2)\Delta$, etc.]

4. $S$ being a quadric cone, vertex $\mathbf{y}$, and $\Pi$ any plane not containing $\mathbf{y}$, $S + \mu\Pi^2 = 0$ $(\mu \neq 0)$ is a non-singular quadric having $S$ as a tangent cone and $\Pi$ as the plane of contact of $\mathbf{y}$.

5. Through any "exterior" line $s$ containing $\mathbf{y}$ (*i.e.* all points of $s$ except $\mathbf{y}$ belong to $\Delta_e$) there pass two tangent planes of $S$. [This plane-pair is the analogue of the tangent cone from any point $\mathbf{z}$ of $s$; its equation is **33** (5) with $\mathbf{z}$ in place of $\mathbf{y}$.] No tangent plane contains any secant or non-secant of $S$.

6. A plane $\Pi$, not containing $\mathbf{y}$, meets a self-polar trihedron in a triangle self-polar w.r.t. the conic in which $\Pi$ meets $S$.

7. In a self-polar trihedron w.r.t. $S$, one edge is an "interior" line, and the other two are "exterior" lines, of $S$.

**Properties in relation to $\Omega$.** (i) $\Omega$ *not containing* $\mathbf{y}$. Here $\Omega$ meets $S$ in a proper conic. Since $\Omega$ has no pole in the usual sense, there is no point $C$ such that every chord through $C$ is bisected at $C$. But it is convenient to define the vertex $\mathbf{y}$ as the centre $C$ of $S$, and to call any line (other than a generator) though $C$ a *diameter*, and in particular to call the edges of a self-polar trihedron a triad of *conjugate diameters*. The reader will then easily verify that the general properties of conjugate diameters of a non-singular quadric are thereby extended to the present case. The properties particular to this case are: Of a triad of conjugate diametral planes, one meets $S$ in $C$ alone and all parallel planes meet $S$ in similar and similarly situated *ellipses*; each of the other two meets $S$ in a line-pair through $C$, and all parallel planes meet $S$ in similar and similarly situated *hyperbolas* with asymptotes parallel to the line-pair. $S$ is called a *quadric cone* in $\mathcal{E}$.

8. There is only one general type of quadric cone in $\mathcal{E}$, and this can be regarded as the transition case between the hyperboloids of one and two sheets.

(ii) $\Omega$ *containing* $\mathbf{y}$. Here $\Omega$ has a polar line w.r.t. $S$. All the generators meet in a point at infinity, and so are parallel, *i.e.* $S$ is a *cylinder*. Three cases arise:

($a$) $\Omega$ meets $S$ in $\mathbf{y}$ alone. Every plane not containing $\mathbf{y}$ meets $\Omega$ in a non-secant, and so meets $S$ in a proper *conic* having no point at infinity, *i.e.* an *ellipse*. $S$ is called an *elliptic cylinder*.

($b$) $\Omega$ meets $S$ in a pair of generators. Every plane not containing $\mathbf{y}$ meets $\Omega$ in a secant, and so meets $S$ in a proper conic having two points at infinity, *i.e.* a *hyperbola*. $S$ is called a *hyperbolic cylinder* and possesses two sheets in $\mathcal{E}$.

In ($a$), ($b$) the polar line of $\Omega$ provides a *line of centres*, which we call the *axis* of the cylinder.

($c$) $\Omega$ is a tangent plane, and so contains its polar line. $S$ has now *no centre*. Every plane not containing $\mathbf{y}$ meets $S$ in a proper conic having one point at infinity, *i.e.* a parabola. $S$ is called a *parabolic cylinder*.

## 36. Reducible Quadrics

**Properties in relation to $\Omega$.** (i) *Plane-pair*. Let the planes be $\Pi$, $\Lambda$ with common line $s$; we have the possibilities: ($a$) $s$ not in $\Omega$; $S$ is a *plane-pair* intersecting in an ordinary line, which may be taken as a line of centres. ($b$) $s$ is in $\Omega$, but $\Pi$, $\Lambda$ are distinct from $\Omega$; $S$ is a pair of *parallel planes*, and the plane midway between them is a plane of centres. ($c$) $\Omega$ is part of $S$, which consists of $\Omega$ and one ordinary plane.

(ii) *Single plane* (counted twice in the equation of $S$). The plane may be ($a$) ordinary, and every point may be taken as a centre, or ($b$) $\Omega$ itself.

(iii) *Single line*. The line may be ($a$) ordinary, and every point may be taken as a centre, or ($b$) a line at infinity.

1. In (i) ($c$), (ii) ($b$) the equation of $S$ is not of the second degree in cartesian coordinates. [In (i) ($c$) the form $S$ has $x_4$ as one factor; in (ii) ($b$) we may take $S \equiv x_4^2$.]

2. If $S$ is the plane-pair $\Pi$, $\Lambda$, with common line $s$, then the polar plane of every point of a plane $\Sigma$ containing $s$ is the harmonic conjugate of $\Sigma$ w.r.t. $\Pi$, $\Lambda$.

## 37. Dual Results

**Non-singular quadric.** Let $\mathsf{A} \equiv [a_{rs}]$ be a non-singular symmetric $4 \times 4$ matrix and $\mathsf{B} \equiv [b_{rs}]$ its reciprocal. Then $[b_{rs}]$ is a non-singular symmetric matrix and (Aitken, **21**)

$$b_{rs} = A_{rs}/\,|\,\mathsf{A}\,|, \quad . \quad . \quad . \quad (1)$$

where $A_{rs}$ is the cofactor of $a_{rs}$ in $|\,\mathsf{A}\,|$.

The locus, assuming it exists, of the equation

$$a_{rs}x_r x_s = 0 \quad . \quad . \quad . \quad (2)$$

is a non-singular quadric $S$. Let $\boldsymbol{\xi}$ be any plane. The pole $\mathbf{y}$ of $\boldsymbol{\xi}$ w.r.t. $S$ is given by **33** (3), *i.e.*, on absorbing the constant $k$ into $\boldsymbol{\xi}$,

$$a_{rs}y_r - \xi_s = 0, \qquad (s = 1, \ldots, 4) \quad (3)$$

and we have seen that (3) has a unique solution. By p. 70 (ii), $\boldsymbol{\xi}$ is a tangent plane if and only if it contains $\mathbf{y}$, *i.e.* $\mathbf{y}$ satisfies also

$$\xi_r y_r = 0. \quad . \quad . \quad . \quad (4)$$

The necessary and sufficient condition for the consistency of (3), (4) is (Aitken, **31**)

$$\begin{matrix} . & \boldsymbol{\xi}' \\ \boldsymbol{\xi} & \mathsf{A} \end{matrix} = 0, \qquad i.e. \qquad A_{rs}\xi_r\xi_s = 0, \quad . \quad (5)$$

or, using (1),

$$b_{rs}\xi_r\xi_s = 0. \quad . \quad . \quad . \quad (6)$$

Therefore *the coordinates of every tangent plane of $S$ satisfy* (6), *and every plane whose coordinates satisfy* (6) *is a tangent plane of $S$.* We call this the *plane* or tangential equation of $S$, as contrasted with the *point* equation (2). Moreover, if $[b_{rs}]$ is *any* non-singular symmetric $4 \times 4$ matrix, (6) is the tangential equation of a non-singular quadric, namely that given by (2) when $[a_{rs}]$ is the reciprocal of $[b_{rs}]$.

By the principle of duality, if we prove a theorem $\mathcal{T}$ concerning the set of *points* $S$ satisfying a general equation of degree two with non-singular matrix, we derive a valid theorem $\mathcal{T}^*$ by substituting "plane," "line," "point" for "point," "line," "plane" in the statement of $\mathcal{T}$. Then $\mathcal{T}^*$ will concern the set of *planes* $\Sigma$ satisfying a general equation of degree two with non-singular matrix. Now $S$ consists of the points of a non-singular quadric, and we have just proved that $\Sigma$ consists of the tangent planes of a non-singular quadric. Hence, corresponding to any theorem about points of a quadric there is a dual about tangent planes of a quadric, and conversely.

The reader should now re-read **33** and note the dual of each result. Owing to the conclusion just reached, he will get the same aggregate of results, but in a different order.

1. The dual of a tangent line of a quadric is a tangent line of a quadric.

2. By (3), the plane-coordinates of the polar plane of $\mathbf{y}$ w.r.t. $S$ are $a_{rs}y_r$ $(s = 1, \ldots, 4)$. Dually, the point-coordinates of the pole of the plane $\boldsymbol{\eta}$ w.r.t. $\Sigma$ are $b_{rs}\eta_r$; these are proportional to $A_{rs}\eta_r$.

3. The centre of $S$ has point-coordinates $A_{41}$, $A_{42}$, $A_{43}$, $A_{44}$. [In (3) take $\boldsymbol{\eta} = (0, 0, 0, 1)$ corresponding to the plane at infinity.]

**Dual of singular quadric.** We might proceed to study the set $\Sigma$ of planes satisfying an equation of the form (6) in cases where $\mathsf{B} \equiv [b_{rs}]$ is singular. There is then no point equation corresponding to (2), since $[b_{rs}]$ has no reciprocal; hence $\Sigma$ is not the set of tangent planes of any non-singular or singular quadric (as we have defined quadric). However, we can foresee the results of such a study by noting that $\Sigma$ is the *dual* of the set $\Sigma^*$ of points satisfying the point equation $b_{rs}x_rx_s = 0$. $\Sigma^*$ is then a singular quadric whose properties are known; the duals of these properties are those of $\Sigma$.

*Rank* $\mathsf{B} = 3$. $\Sigma^*$ is a quadric cone; it may be taken to be a tangent cone of a non-singular quadric $Q^*$ (**35** 4), *i.e.* $\Sigma^*$ consists of all points on the tangent lines of $Q^*$ which pass

TABLE 2

**Central Quadric**—Unique Centre

| Matrix A | Quadric $S$ | Sections by conjugate diametral planes | Generators | Standard equation |
|---|---|---|---|---|
| $\lvert \mathsf{A} \rvert \neq 0, A_{44} \neq 0$ (Rank A = 4) | **Ellipsoid** | Ellipses | None | $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2 = 1$ |
| | **Hyperboloid of one sheet** | Two hyperbolas; one ellipse | Two systems | $x^2/\alpha^2+y^2/\beta^2-z^2/\gamma^2 = 1$ |
| | **Hyperboloid of two sheets** | Two hyperbolas; one plane not meeting $S$ | None | $x^2/\alpha^2-y^2/\beta^2-z^2/\gamma^2 = 1$ |
| $\lvert \mathsf{A} \rvert = 0, A_{44} \neq 0$ (Rank A = 3) | **Single point** | Single point | None | $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2 = 0$ |
| | **Cone** | Two line-pairs; one single point | One system | $x^2/\alpha^2+y^2/\beta^2-z^2/\gamma^2 = 0$ |

**Central Quadric**—Line of Centres

| Matrix A | Quadric $S$ | Sections by diametral and non-diametral plane | Generators | Standard equation |
|---|---|---|---|---|
| $\lvert \mathsf{A} \rvert = 0, A_{44} = 0$ Rank A = 3 | **Elliptic cylinder** | Two parallel lines; ellipse | Parallel generators | $x^2/\alpha^2+y^2/\beta^2 = 1$ |
| | **Hyperbolic cylinder** | Two parallel lines; hyperbola | Parallel generators | $x^2/\alpha^2-y^2/\beta^2 = 1$ |
| Rank A = 2 | **Single line** | Single line; single point | Single generator | $x^2/\alpha^2+y^2/\beta^2 = 0$ |
| | **Pair of non-parallel planes** | Single line; pair of non-parallel lines | All lines in each plane | $x^2/\alpha^2-y^2/\beta^2 = 0$ |

**Central Quadric**—Plane of Centres

| Matrix A | Quadric $S$ | Section by non-diametral plane | Generators | Standard equation |
|---|---|---|---|---|
| Rank A $= 2$ | **Pair of parallel planes** | Pair of parallel lines | All lines in each plane | $x^2 = \alpha^2$ |
| Rank A $= 1$ | **Single plane** (counted twice) | Single line | All lines in plane | $x^2 = 0$ |

**Non-central Quadric**

| Matrix A | Quadric $S$ | Sections by diametral and non-diametral plane | Generators | Standard equation |
|---|---|---|---|---|
| $\lvert A \rvert \neq 0, A_{44} = 0$ (Rank A $= 4$) | **Elliptic paraboloid** | Parabola; ellipse, single point, or no intersection | None | $x^2/\alpha^2 + y^2/\beta^2 = 2z/\gamma$ |
| | **Hyperbolic paraboloid** | Parabola; hyperbola | Two systems | $x^2/\alpha^2 - y^2/\beta^2 = 2z/\gamma$ |
| $\lvert A \rvert = 0, A_{44} = 0$ Rank A $= 3$ | **Parabolic cylinder** | Single line; parabola | Parallel generators | $x^2 = 2\delta y$ |

through a fixed point. Dually, $\Sigma$ consists of all planes through the tangent lines of a non-singular quadric $Q$ which lie in a fixed plane. Hence $\Sigma$ *consists of the aggregate of planes containing the tangent lines of a proper conic.* These lines may be called generators of $\Sigma$; then every plane of $\Sigma$ contains one and only one generator, except the plane of the conic; the latter plane contains all the generators and may be called a singular plane of $\Sigma$.

*Rank* $\mathsf{B} = 2$. Here $\Sigma^*$ and so $\Sigma$ is reducible. Therefore we can write its equation in the form, say, $(u_r\xi_r)(v_s\xi_s) = 0$, and then $\Sigma$ consists of the aggregate of planes $\boldsymbol{\xi}$ satisfying $u_r\xi_r = 0$ or $v_s\xi_s = 0$. These are the plane-equations of two points $\mathbf{u}, \mathbf{v}$. Thus $\Sigma$ *consists of the aggregate of planes containing* $\mathbf{u}$ *or* $\mathbf{v}$, just as $\Sigma^*$ consists of the aggregate of points contained in two planes.

*Rank* $\mathsf{B} = 1$. $\Sigma$ *consists of the aggregate of planes through a single point,* counted twice in the equation of $\Sigma$.

## 38. Classification of Quadrics in $\mathcal{E}$

We can now extract from **34–36** the classification of quadrics according to their properties in $\mathcal{E}$ given in Table 2. The last column gives the standard cartesian equation from Chapter VI.

CHAPTER VI

# QUADRIC IN CARTESIAN COORDINATES; STANDARD FORMS

## 39. Algebraic Lemmas

The results of this section, in the form required, are not readily accessible in textbooks of elementary algebra; their derivation here may therefore assist the student.

I. **Discriminating cubic.*** Let us consider the matrices

$$\mathsf{D} \equiv \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \qquad \mathsf{D}_\lambda \equiv \begin{bmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{bmatrix}$$

where $a$, $b$, $c$, $f$, $g$, $h$ are given real numbers, and write

$$D \equiv |\mathsf{D}|, \qquad D_\lambda \equiv |\mathsf{D}_\lambda| .$$

This is in accord with the practice we shall adopt of denoting the cofactors of $a, h, \ldots, d$ in $\Delta$ by $A, H, \ldots, D$, where

$$\Delta \equiv |\mathsf{S}| \qquad \mathsf{S} \equiv \begin{matrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{matrix} \tag{1}$$

We shall denote the cofactors of $a, \ldots, h$ in $D$ by $\mathscr{A}, \ldots, \mathscr{H}$, and the cofactors of $a-\lambda, \ldots, h$ in $D_\lambda$ by $\mathscr{A}_\lambda, \ldots, \mathscr{H}_\lambda$. In the application required, the equation $D_\lambda = 0$, which is in fact †

$$\lambda^3-(a+b+c)\lambda^2+(\mathscr{A}+\mathscr{B}+\mathscr{C})\lambda-D = 0, \quad . \tag{2}$$

will be called the *discriminating cubic* of the quadratic form having matrix $\mathsf{D}$.

* The discussion is based on Turnbull and Aitken, *Canonical Matrices* (1932), 101, Ex. 1.

† The l.h.s. is actually $-D_\lambda$.

Suppose first that $f, g, h \neq 0$. Let $\lambda = \gamma_1, \gamma_2$ be the roots of

$$\mathscr{C}_\lambda \equiv (a-\lambda)(b-\lambda)-h^2 = 0. \quad . \quad . \quad (3)$$

Then $\gamma_1, \gamma_2$ are real, and $\gamma_1 < a,\ b < \gamma_2$.

By Jacobi's theorem, or by direct verification,

$$(a-\lambda)D_\lambda \equiv \mathscr{B}_\lambda \mathscr{C}_\lambda - \mathscr{F}_\lambda^2. \quad . \quad . \quad (4)$$

Also $\mathscr{F}_\lambda \equiv \mathscr{F}+\lambda f$ and $f \neq 0$; therefore $\mathscr{F}_\lambda$ vanishes for one and only one value of $\lambda$.

Case (i). $\mathscr{F}_{\gamma_1} \neq 0$, $\mathscr{F}_{\gamma_2} \neq 0$. Put $\lambda = \gamma_1$ in (4); since $a-\gamma_1 > 0$, $\mathscr{C}_{\gamma_1} = 0$, $\mathscr{F}_{\gamma_1}^2 > 0$, we obtain $D_{\gamma_1} < 0$. Analogously, $D_{\gamma_2} > 0$. Therefore, when $\lambda = -\infty, \gamma_1, \gamma_2, \infty$, the sign of $D_\lambda$ is $+, -, +, -$. Thus (2) *has three real and distinct roots* $\lambda_1, \lambda_2, \lambda_3$ *such that* $\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \lambda_3$.

Case (ii). $\mathscr{F}_{\gamma_1} = 0$, $\mathscr{F}_{\gamma_2} \neq 0$. Put $\lambda = \gamma_1$ in (4); since $a-\gamma_1 > 0$, $\mathscr{C}_{\gamma_1} = 0$, $\mathscr{F}_{\gamma_1} = 0$, we obtain $D_{\gamma_1} = 0$. As in (i), $D_{\gamma_2} > 0$. Therefore $D_\lambda$ has the *same* sign when $\lambda = -\infty, \gamma_2$, and has one zero $\gamma_1$ between these values; consequently it has a second zero between them. As in (i), there is a third zero between $\lambda = \gamma_2, \infty$. Thus (2) has three real roots, $\lambda_1, \lambda_2, \lambda_3$, two at least being distinct, such that $\lambda_1 = \gamma_1 < \gamma_2$; $\lambda_2 < \gamma_2 < \lambda_3$.

Now suppose (2) has a double root; from what has just been proved, this is necessarily $\lambda_1 (= \gamma_1)$. Then in (4) $\lambda_1$ is a double zero of $D_\lambda$ and of $\mathscr{F}_\lambda^2$, and so is a double zero of $\mathscr{B}_\lambda \mathscr{C}_\lambda$. Therefore, since $\lambda_1 (= \gamma_1)$ is only a simple zero of $\mathscr{C}_\lambda$, it must be a zero also of $\mathscr{B}_\lambda$. Using these properties in further relations like (4), we can show that $\lambda_1$ is a zero also of $\mathscr{A}_\lambda, \mathscr{G}_\lambda, \mathscr{H}_\lambda$. Thus *a double zero of* $D_\lambda$ *is a zero of every cofactor of* $D_\lambda$.

It is easy to prove the converse. We can, in fact, prove somewhat more, viz., *if a value* $\lambda_1$ *of* $\lambda$ *is a zero of the three cofactors* $\mathscr{F}_\lambda, \mathscr{G}_\lambda, \mathscr{H}_\lambda$, *then it is necessarily a zero also of* $\mathscr{A}_\lambda, \mathscr{B}_\lambda, \mathscr{C}_\lambda$ *and consequently*, from (4), *a double zero of* $D_\lambda$ (for $a-\lambda_1 \neq 0$). Since then the remaining zero, say $\lambda_3$, of $D_\lambda$ must be simple, it cannot make $\mathscr{F}_\lambda, \mathscr{G}_\lambda, \mathscr{H}_\lambda$ all vanish.

Case (iii). $\mathscr{F}_{\gamma_1} \neq 0$, $\mathscr{F}_{\gamma_2} = 0$. This is similar to (ii).

A little elaboration of these arguments extends the results to cases where $f, g, h$ are not all different from zero.

It follows that, if $\gamma_1 \neq \gamma_2$, $D_\lambda$ cannot have *three* equal zeros. So if it has zeros $\lambda_1 = \lambda_2 = \lambda_3$, then certainly $\gamma_1 = \gamma_2$. From

(3), this requires $(a+b)^2-4(ab-h^2) = 0$, *i.e.* $(a-b)^2+4h^2 = 0$, giving $a = b$, $h = 0$. Similarly, it is necessary that $b = c$, $f = 0$; $c = a$, $g = 0$. But if these are satisfied, then $D_\lambda \equiv (a-\lambda)^3$ and so $\lambda_1 = \lambda_2 = \lambda_3 = a$. Thus $D_\lambda$ *has a triple zero* $\lambda_1$ *if and only if* $a = b = c(= \lambda_1)$, $f = g = h = 0$, *i.e.* $D_{\lambda_1}$ *has every element zero.*

We can restate the results thus: $D_\lambda = 0$ *has three real roots* $\lambda_1, \lambda_2, \lambda_3$ *such that*

(a) $\lambda_1 \neq \lambda_2 \neq \lambda_3$ *if and only if* $D_{\lambda_1}, D_{\lambda_2}, D_{\lambda_3}$ *have rank* 2;
(b) $\lambda_1 = \lambda_2 \neq \lambda_3$ *if and only if* $D_{\lambda_1}$ *has rank* 1, $D_{\lambda_3}$ *has rank* 2;
(c) $\lambda_1 = \lambda_2 = \lambda_3$ *if and only if* $D_{\lambda_1}$ *has rank* 0.

II. **Principal directions.** Consider now the equations

$$\left.\begin{aligned} al+hm+gn &= \lambda l, \\ hl+bm+fn &= \lambda m, \\ gl+fm+cn &= \lambda n. \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

These have a solution in $l$, $m$, $n$ (not all zero) if and only if $D_\lambda = 0$, *i.e.* $\lambda = \lambda_1, \lambda_2, \lambda_3$.

Let a solution when $\lambda = \lambda_1$ be $l_1$, $m_1$, $n_1$, and suppose these to be normalised so that they can be regarded as direction-cosines; similarly for $\lambda_2$, $\lambda_3$. Then writing $\lambda = \lambda_1$, and so $l = l_1$, etc., in (5) and multiplying the equations by $l_2$, $m_2$, $n_2$, respectively, we get

$$\begin{aligned} \lambda_1(l_1l_2+m_1m_2+n_1n_2) = al_1l_2+bm_1m_2+cn_1n_2+f(m_1n_2+m_2n_1) \\ +g(n_1l_2+n_2l_1)+h(l_1m_2+l_2m_1). \end{aligned} \qquad (6)$$

From the symmetry of the r.h.s. it follows that

$$\lambda_1(l_1l_2+m_1m_2+n_1n_2) = \lambda_2(l_1l_2+m_1m_2+n_1n_2).$$

Hence, if $\lambda_1 \neq \lambda_2$, we have $l_1l_2+m_1m_2+n_1n_2 = 0$, *i.e. directions corresponding to unequal zeros of* $D_\lambda$ *are orthogonal.*

If $\lambda_1$ is a simple zero of $D_\lambda$, then, from I, equations (5) have rank 2 when $\lambda = \lambda_1$, and so have a unique solution. If $\lambda_1(= \lambda_2)$ is a double zero, then (5) have rank 1 when $\lambda = \lambda_1$, and so impose only one condition on $l_1$, $m_1$, $n_1$, which must then be equivalent to the condition that $(l_1, m_1, n_1)$ is orthogonal to $(l_3, m_3, n_3)$. If $\lambda_1$ is a triple zero, then (5) have rank 0 when $\lambda = \lambda_1$, and so impose no condition on $l_1$, $m_1$, $n_1$. Hence we have the following cases corresponding to those in I:

(a) $\lambda_1 \neq \lambda_2 \neq \lambda_3$. $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ *form a unique triad of orthogonal directions.*

(b) $\lambda_1 = \lambda_2 \neq \lambda_3$. $(l_3, m_3, n_3)$ *is unique;* $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$ *can be any directions orthogonal to* $(l_3, m_3, n_3)$, *and we shall take them so as to be orthogonal also to each other.*

(c) $\lambda_1 = \lambda_2 = \lambda_3$. $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ *can be any directions, and we shall take them so as to be orthogonal to each other.*

In each case we call the directions *principal directions* of $\mathsf{D}$.

III. **Canonical form.** Now let $Q$ be the quadratic form in rectangular coordinates $x$, $y$, $z$, whose matrix is $\mathsf{D}$, *i.e.*

$$Q = ax^2+by^2+cz^2+2fyz+2gzx+2hxy. \quad . \quad (7)$$

Rotate the axes to the principal directions of $\mathsf{D}$ found in II. If $x'$, $y'$, $z'$ are the new coordinates, we have from **5** (6)

$$[xyz] = [x'y'z']\mathsf{T}, \quad \text{where} \quad \mathsf{T} \equiv \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}. \quad (8)$$

Hence the matrix of $Q$ is transformed to $\mathsf{TDT'}$, where $\mathsf{T'}$ is the transpose of $\mathsf{T}$ (see Aitken, **10 11**).

Now

$$\mathsf{DT'} = \begin{bmatrix} al_1+hm_1+gn_1 & al_2+hm_2+gn_2 & al_3+hm_3+gn_3 \\ hl_1+bm_1+fn_1 & hl_2+bm_2+fn_2 & hl_3+bm_3+fn_3 \\ gl_1+fm_1+cn_1 & gl_2+fm_2+cn_2 & gl_3+fm_3+cn_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 l_1 & \lambda_2 l_2 & \lambda_3 l_3 \\ \lambda_1 m_1 & \lambda_2 m_2 & \lambda_3 m_3 \\ \lambda_1 n_1 & \lambda_2 n_2 & \lambda_3 n_3 \end{bmatrix},$$

using (5) with $\lambda = \lambda_1, \lambda_2, \lambda_3$. Therefore $\mathsf{TDT'} =$

$$\begin{bmatrix} \lambda_1(l_1^2+m_1^2+n_1^2) & \lambda_2(l_1l_2+m_1m_2+n_1n_2) & \lambda_3(l_1l_3+m_1m_3+n_1n_3) \\ \lambda_1(l_2l_1+m_2m_1+n_2n_1) & \lambda_2(l_2^2+m_2^2+n_2^2) & \lambda_3(l_2l_3+m_2m_3+n_2n_3) \\ \lambda_1(l_3l_1+m_3m_1+n_3n_1) & \lambda_2(l_3l_2+m_3m_2+n_3n_2) & \lambda_3(l_3^2+m_3^2+n_3^2) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \cdot & \cdot \\ \cdot & \lambda_2 & \cdot \\ \cdot & \cdot & \lambda_3 \end{bmatrix},$$

using **5** (2), (3). Consequently

$$Q = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2, \qquad (9)$$

giving the *canonical form* of $Q$. The algebra can be reversed to show that conversely, if $Q$ has canonical form in rectangular coordinates, the axes are along principal directions of $Q$. Thus, in rectangular axes and apart from the order of $\lambda_1$, $\lambda_2$, $\lambda_3$, *the canonical form is unique.*

1. Verify (9) by direct substitution of **5** (6) in (7), using (6) and similar relations.

IV. **Invariants.** Since the canonical form is unique, all possible forms of $Q$ like (7) are derivable from the canonical form by some orthogonal transformation. Now, $\lambda_1$, $\lambda_2$, $\lambda_3$ being the roots of (2), we have

$$a+b+c = \lambda_1+\lambda_2+\lambda_3, \qquad (10)$$

$$\mathcal{A}+\mathcal{B}+\mathcal{C} \equiv bc+ca+ab-f^2-g^2-h^2 = \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2, \qquad (11)$$

$$D \equiv abc-af^2-bg^2-ch^2+2fgh = \lambda_1\lambda_2\lambda_3. \qquad (12)$$

Therefore the functions of $a, b, \ldots, h$ on the l.h.s. of (10)–(12) depend only on the canonical form of $Q$, and so are *invariant* for all orthogonal transformations of $Q$. Moreover $Q$ has no other invariants independent of these; for if these are given, then $\lambda_1$, $\lambda_2$, $\lambda_3$, and consequently the canonical form of $Q$, are completely determined.

Consider further the expression

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d,$$

where $a, b, \ldots, d$ are given (real) constants. This may be regarded as a quadratic form in $x, y, z, t$, where $t \equiv 1$, with matrix $\mathsf{S}$ given by (1).

The transformation (8) may now be written:

$$[xyzt] = [x'y'z't']\,\mathsf{H}, \quad \text{where} \quad \mathsf{H} = \begin{bmatrix} \mathsf{T} & \cdot \\ \cdot & 1 \end{bmatrix}, \quad \text{and} \quad t = t' = 1.$$

Under this transformation $S$ becomes a quadratic form in $x', y', z', t'$ with matrix $\mathsf{HSH'}$, where $\mathsf{H'}$ is the transpose of $\mathsf{H}$. Now $|\mathsf{H}| = |\mathsf{H'}| = |\mathsf{T}| = 1$, from **5** 4. Therefore

$$|\mathsf{HSH'}| = |\mathsf{H}|\,|\mathsf{S}|\,|\mathsf{H'}| = |\mathsf{S}| = \Delta,$$

*i.e.* $\Delta$ is invariant under a rotation of axes.

Further, the change of origin of parallel axes **5** (1) may be written:

$$[xyzt] \rightleftharpoons [x^*y^*z^*t^*]\,\mathsf{K},$$

where $\mathsf{K} = \begin{bmatrix} & & & \\ & & & \\ & & 1 & \\ \xi & \eta & \zeta & 1 \end{bmatrix}$ and $t = t^* = 1.$

Since $|\,\mathsf{K}\,| = 1$, it follows as before that $\Delta$ is invariant under a change of origin.

Combining these results, we have: *$\Delta$ is invariant under any change of rectangular axes.*

## 40. Quadric in Cartesian Coordinates

Using rectangular cartesian coordinates, the general equation of the second degree considered in **30** becomes

$$S(x, y, z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy \\ +2ux+2vy+2wz+d = 0, \qquad (1)$$

where we have replaced the coefficients $a_{11}, a_{12}, a_{13}, a_{14}, \ldots$ by $a, h, g, u, \ldots$ We continue to denote the determinant of these coefficients by $\Delta$, but shall denote the cofactors of $a, h, \ldots, d$ in $\Delta$ by $A, H, \ldots, D$, instead of $A_{11}, A_{12}, \ldots, A_{44}$.

Using **5** (1), change the origin in (1) to $(\xi, \eta, \zeta)$ and let $x, y, z$ now denote the *new* coordinates. Then (1) becomes

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2(a\xi+h\eta+g\zeta+u)x \\ +2(h\xi+b\eta+f\zeta+v)y+2(g\xi+f\eta+c\zeta+w)z+S(\xi, \eta, \zeta) = 0. \quad (2)$$

Now let $\xi, \eta, \zeta$ be chosen, *if possible*, to satisfy the equations

$$\left.\begin{aligned} a\xi+h\eta+g\zeta+u &= 0, \\ h\xi+b\eta+f\zeta+v &= 0, \\ g\xi+f\eta+c\zeta+w &= 0. \end{aligned}\right\} \quad . \quad . \quad (3)$$

We then have

$$\begin{aligned} S(\xi, \eta, \zeta) &\equiv \xi(a\xi+h\eta+g\zeta+u)+\eta(h\xi+b\eta+f\zeta+v) \\ &\quad +\zeta(g\xi+f\eta+c\zeta+w)+u\xi+v\eta+w\zeta+d \\ &= u\xi+v\eta+w\zeta+d \equiv d', \quad \text{say.} \end{aligned} \qquad (4)$$

So (2) becomes

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d' = 0. \qquad (5)$$

Then, if $(x, y, z)$ satisfies (5), so does $(-x, -y, -z)$, *i.e.* $S$ is symmetrical about the new origin, which is therefore a *centre* of $S$.

1. Translating **33** (2), the polar plane of an ordinary point $(x_1, y_1, z_1)$ w.r.t. $S$ is

$$(ax_1+hy_1+gz_1+u)x+(hx_1+by_1+fz_1+v)y+(gx_1+fy_1+cz_1+w)z +ux_1+vy_1+wz_1+d = 0. \qquad (6)$$

2. (6) is the plane at infinity if the coefficients of $x$, $y$, $z$ vanish; so we recover (3) as the equations for the centre as defined in **34**.

3. The polar plane of a point at infinity $(l, m, n, 0)$ is

$$(al+hm+gn)x+(hl+bm+fn)y+(gl+fm+cn)z+ul+vm+wn = 0. \qquad (7)$$

In accordance with **27, 34** this is the diametral plane conjugate to the direction $(l, m, n)$.

## 41. Central Quadrics

We have the following cases in which equations **40** (3) are consistent.

$D \neq 0$ Here (3) have the unique solution $(\xi, \eta, \zeta) = (U/D,\ V/D,\ W/D)$. Hence from **40** (4)

$$d' = (uU+vV+wW+dD)/D = \Delta/D. \qquad (1)$$

Now rotate the axes to the principal directions of $Q$, using **39** III, so that **40** (5) becomes

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+\Delta/D = 0, \qquad (2)$$

where $x$, $y$, $z$ refer to the new axes. Here, by **39** (12), $\lambda_1\lambda_2\lambda_3 = D \neq 0$, giving $\lambda_1, \lambda_2, \lambda_3 \neq 0$.

1. The determinant of the coefficients in (2) is $\lambda_1\lambda_2\lambda_3\Delta/D = \Delta$, thus verifying the invariance of $\Delta$.

2. *The axes in* (2) *are conjugate diameters.* [For, by **40** (7),

the plane conjugate to the direction (1, 0, 0) is $\lambda_1 x = 0$, etc.] These diameters are called the *principal axes* of the quadric.

3. *The only directions perpendicular to their conjugate planes are the principal directions.* [For, by **40** (7), the plane conjugate to $(l, m, n)$ is $\lambda_1 lx + \lambda_2 my + \lambda_3 nz = 0$.] This is perpendicular to $(l, m, n)$ if and only if

$$\lambda_1 l/l = \lambda_2 m/m = \lambda_3 n/n. \quad . \quad . \quad . \qquad (3)$$

If $\lambda_1 \neq \lambda_2 \neq \lambda_3$, (3) are satisfied only if two of $l, m, n$ are zero, *i.e.* if $(l, m, n)$ is the direction of one of the axes, and these are the unique principal directions (**39** II).

If $\lambda_1 = \lambda_2 \neq \lambda_3$, (3) are satisfied only if either $l = m = 0$, or $n = 0$, *i.e.* if $(l, m, n)$ is the $z$-direction, or any perpendicular direction, and these give the principal directions (**39** II).

If $\lambda_1 = \lambda_2 = \lambda_3$, (3) are satisfied for all $(l, m, n)$, and here every direction is a principal direction (**39** II).

$D \neq 0, \Delta \neq 0 \,|$. If $\Delta \neq 0$, equation (2) may be reduced to one of the following forms which include all possible combinations of signs of the coefficients (after, if necessary, a permutation of $x, y, z$). The identification of the surfaces follows from Table 2, remembering that, from 2, the co-ordinate planes are conjugate diametral planes.

(i) $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2} = 1.$ This is an *ellipsoid*, since the sections by $x = 0$, $y = 0$, $z = 0$ are ellipses.

(ii) $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}-\dfrac{z^2}{\gamma^2} = 1.$ This is a *hyperboloid of one sheet*, since the sections by $x = 0, y = 0$ are hyperbolas, that by $z = 0$ an ellipse.

(iii) $= 1.$ This is a *hyperboloid of two sheets*, since $x = 0$ does not meet $S$, and the sections by $y = 0, z = 0$ are hyperbolas.

(iv) $x^2 \quad -\dfrac{y^2}{\beta^2}-\dfrac{z^2}{\gamma^2} = 1.$ This equation has *no locus*.

Note that the general appearance * of these surfaces can be gathered from particular cases in which they are surfaces of revolution. (i) If $\beta = \gamma$, the surface is given by rotating the ellipse $x^2/\alpha^2+y^2/\beta^2 = 1$, $z = 0$, about the $x$-axis. (ii) If $\alpha = \beta$, the surface is given by rotating the hyperbola $x^2/\alpha^2-z^2/\gamma^2 = 1$, $y = 0$, about its "conjugate axis." (iii) If $\beta = \gamma$, the surface is given by rotating the same hyperbola about its "transverse axis."

$D \neq 0, \Delta = 0 \,\big|$. If $\Delta = 0$, equation (2) may be reduced to one of the following forms (after, if necessary, permuting $x$, $y$, $z$):

(v) $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2} = 0$. This gives the *single point* (0, 0, 0).

(vi) $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}-\dfrac{z^2}{\gamma^2} = 0$. This is a *cone*, since the sections by $x = 0$, $y = 0$ are line-pairs, that by $z = 0$ the single point (0, 0, 0) (*cf.* Table 2).

$D = 0$; rank $\mathsf{D} = 2$ If rank $\mathsf{D} = 2$, equations **40** (3) have no solution unless their rank is 2, *i.e.* $U, V, W, D = 0$, so that also $\Delta = 0$, in which case they have a line of solutions given by any two of the equations (Aitken, **30**). Let $(\xi, \eta, \zeta)$ be any solution and $d'$ be given by **40** (4); then, eliminating $\xi$, $\eta$ from

$$\begin{aligned} a\xi+h\eta+g\zeta+u &= 0, \\ h\xi+b\eta+f\zeta+v &= 0, \\ u\xi+v\eta+w\zeta+d-d' &= 0, \end{aligned}$$

we obtain

$$\begin{vmatrix} a & & g\zeta+u \\ h & & f\zeta+v \\ u & v & w\zeta+d-d' \end{vmatrix} = 0, \quad \textit{i.e.} \quad W\zeta+C-d'\mathscr{C} = 0,$$

* Pictures of quadric surfaces are given in some larger books, and models are available in many mathematical departments.

giving, since $W = 0$,

$$d' = C/\mathcal{C}.$$

Similarly we find

$$d' = A/\mathcal{A} = B/\mathcal{B} = \ldots = H/\mathcal{H}, \qquad . \qquad (4)$$

showing, incidentally, that $A/\mathcal{A}$, etc., are invariants in this case.

Now $\lambda_1\lambda_2\lambda_3 = D = 0$, so at least one of $\lambda_1$, $\lambda_2$, $\lambda_3$, say $\lambda_3$, is zero; also $\lambda_3$ is then not a repeated root, since otherwise $\mathsf{D}_{\lambda_3} \equiv \mathsf{D}$ would have rank $< 2$, by **39** I. Hence $\lambda_1, \lambda_2 \neq 0$. Rotating the axes to the principal directions of $Q$, **40** (5) therefore becomes

$$\lambda_1 x^2 + \lambda_2 y^2 + d' = 0, \qquad . \qquad . \qquad (5)$$

where $x$, $y$, $z$ refer to the new axes, and $d'$ is given by (4).

If $A$, $B$, $C$, $F$, $G$, $H$ are not all zero, then $d' \neq 0$ and (5) may be reduced to one of the following forms (if necessary, interchanging $x$, $y$):

(vii) $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$. This is an *elliptic cylinder*, since all sections parallel to $z = 0$ are similarly situated equal ellipses.

(viii) $\dfrac{x^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 1$. This is a *hyperbolic cylinder*, for an analogous reason.

(ix) $-\dfrac{x^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 1$. This equation has *no locus*.

If $A$, $B$, $C$, $F$, $G$, $H$ all vanish, then $d' = 0$ and (5) may be reduced to one of the forms:

(x) $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 0$. This gives the *single line* $x = 0$, $y = 0$.

(xi) $\dfrac{x^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 0$. This is the *pair of intersecting planes* $x/\alpha \pm y/\beta = 0$.

$D = 0$; rank $\mathsf{D} = 1$ |. If rank $\mathsf{D} = 1$ equations **40** (3) have no solution unless their rank is 1, in which case they have a plane of solutions given by any one of the equations (Aitken, **30**). Let $(\xi, \eta, \zeta)$ be any solution, and let $d'$ be given by **40** (4).

Since rank $\mathsf{D} = 1$, by **39** I two of $\lambda_1$, $\lambda_2$, $\lambda_3$ are zero and one, say $\lambda_3$, is not zero. Therefore, on rotating to principal directions, **40** (5) becomes

$$\lambda_3 z^2 + d' = 0. \qquad . \qquad . \qquad . \qquad (6)$$

If $d' \neq 0$, this may be reduced to one of the forms:

(xii) $z^2 = \gamma^2$. This is the *pair of parallel planes* $z = \pm\gamma$.

(xiii) $z^2 = -\gamma^2$. This equation has *no locus*.

If $d' = 0$, (6) reduces to

(xiv) $z^2 = 0$. This is the *single plane* $z = 0$, counted twice.

4. Derive values of $d'$ analogous to (4).

$D = 0$; rank $\mathsf{D} = 0$ |. If rank $\mathsf{D} = 0$, all the elements of $D$ are zero, and all terms of the second degree in **40** (1) have zero coefficients.

## 42. Non-central Quadrics

It remains to consider cases in which **40** (3) have no solution, so that we cannot choose $\xi$, $\eta$, $\zeta$ so as to make the terms of the first degree in **40** (2) all zero. We shall therefore first rotate the axes to the principal directions of $Q$ *without change of origin*. Letting $x$, $y$, $z$ refer to the new axes, **40** (1) then becomes

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + 2u'x + 2v'y + 2w'z + d = 0, \qquad (1)$$

where $u'$, $v'$, $w'$ are new constants. The following will be found to give the cases not already disposed of:

$D = 0$; rank $\mathsf{D} = 2$; $\Delta \neq 0$ |. If rank $\mathsf{D} = 2$, then as before $\lambda_1$, $\lambda_2$ (say) $\neq 0$, $\lambda_3 = 0$. So (1) may be written

$$\lambda_1(x+u'/\lambda_1)^2+\lambda_2(y+v'/\lambda_2)^2+2w'\{z+(d-u'^2/\lambda_1-v'^2/\lambda_2)/2w'\} = 0,$$

showing that, by a change of origin, it becomes in new $x$, $y$, $z$,

$$\lambda_1 x^2+\lambda_2 y^2+2w'z = 0. \qquad (2)$$

Since, by **39** IV, $\Delta$ is invariant, we have

$$\Delta = \lambda_1 \qquad = -\lambda_1\lambda_2 w'^2, \text{ giving}$$

$$w' \qquad w' \qquad w' = \pm\sqrt{-\frac{\Delta}{\lambda_1\lambda_2}}. \qquad (3)$$

Note that $w'$ is necessarily real; the ambiguity of sign merely corresponds to the two possible choices of the sense of the $z$-axis. (2) may now be written in one of the following forms, according as $\lambda_1$, $\lambda_2$ have the same or opposite signs. The identification follows from Table 2, using 1 below.

(xv) $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}=\dfrac{2z}{\gamma}$. This is an *elliptic paraboloid*, since the sections by $x = 0$, $y = 0$ are parabolas, that by $z = k\gamma$ (with $k > 0$) an ellipse.

(xvi) $\dfrac{x^2}{\alpha^2}-\dfrac{y^2}{\beta^2}=\dfrac{2z}{\gamma}$. This is a *hyperbolic paraboloid*, since the sections by $x = 0$, $y = 0$ are parabolas, that by $z = k\gamma$ a hyperbola.

The origin in (xv), (xvi) is called the *vertex* of the paraboloid.

$D = 0$; rank $\mathsf{D} = 1$; $\Delta = 0$; rank $\mathsf{S} = 3$ |. If rank $\mathsf{D} = 1$, then as before $\lambda_1$, $\lambda_2$ (say) $= 0$, $\lambda_3 \neq 0$. Also $u'$, $v'$ are not both zero; otherwise (1) would reduce to a pair of parallel planes. So (1) may be written, if $v'$ (say) $\neq 0$, $\lambda_3(z+w'/\lambda_3)^2+2u'x+2v'\{y+(d-w'^2/\lambda_3)/2v'\} = 0$, showing that, by a change of origin, it becomes in new $x$, $y$, $z$,

$$\lambda_3 z^2+2u'x+2v'y = 0. \qquad . \qquad . \qquad (4)$$

If $u' \neq 0$, we may, by a rotation of axes in which $OZ$ is kept fixed, replace $u'x+v'y$ by $v''y$ in new $x$, $y$, and so replace (4) by $\lambda_3 z^2+2v''y = 0$, or (say)

(xvii) $z^2 = 2\delta y$. This is a *parabolic cylinder*, since all sections parallel to $x = 0$ are similarly situated equal parabolas.

1. If $\lambda_1$, $\lambda_2 \neq 0$, $\lambda_3 = 0$, any plane parallel to $OZ$ is a diametral plane of (1); if $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, any plane parallel to $z = 0$ is a diametral plane. [Use (1) and **40** (7).]

2. If $\alpha^2 = \beta^2$, (xv) is given by rotating the parabola $x^2/\alpha^2 = 2z/\gamma$, $y = 0$ about its axis. There is no surface of revolution which is a hyperbolic paraboloid.

3. *$S$ being real and non-singular, it is a ruled quadric if and only if $\Delta > 0$.*

4. Verify that **41**, **42** do in fact give an exhaustive classification of the algebraic possibilities.

## 43. Numerical Examples

Given an equation of the form **40** (1) with numerical coefficients, consider the problem of discovering the type of the quadric $S$ and reducing its equation to the appropriate one of the standard forms (i)–(xvii). Chapter V shows that the type of $S$ can be found by examination of the matrix of its coefficients (and, since this matrix is symmetric, properties like those in Aitken, **28** 2, **30** 5, facilitate the work). But the methods of the present chapter are usually preferable in practical cases. The precise procedure depends upon the amount of information sought, *e.g.* whether only the standard

equation, or also the description of the final in terms of the initial coordinate system, is demanded. In the main, it is probably best to follow the steps of the general theory in **41, 42**. But one naturally looks first for any obvious special features of the given equation, *e.g.* it may have obvious factors (in which case $S$ is a pair of planes whose equations may be written down at once), or the terms of the second degree may form a perfect square (when we must have one of the cases (xii), (xiii), (xiv), or (xvii), and the standard form can be found directly). The following examples illustrate the chief features of the problem, but the reader should work many more, either constructed by himself or taken from other sources.

In practical examples we keep $x$, $y$, $z$ for the original coordinates, and use $X$, $Y$, $Z$ for those giving the standard form.

1. $7x^2-8y^2-8z^2-2yz-8zx+8xy-16x+14y-14z-5 = 0.$

The equations **40** (3) for the centre are

$$\begin{aligned} 7\xi+4\eta-4\zeta-8 &= 0,\\ 4\xi-8\eta-\zeta+7 &= 0,\\ -4\xi-\eta-8\zeta-7 &= 0, \end{aligned}$$

giving the unique solution $\xi = 0$, $\eta = 1$, $\zeta = -1$. Hence from **40** (4),

$$d' = -8\xi+7\eta-7\zeta-5 = 9.$$

The discriminating cubic is

$$\begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0,$$

having roots $-9$, $-9$, $9$. Hence by **41** (2) the quadric referred to its axes is $-9X^2-9Y^2+9Z^2+9 = 0$, *i.e.*

$$X^2+Y^2-Z^2 = 1.$$

It is therefore by **41** (ii) a *hyperboloid* (*of revolution*) *of one sheet.*

It will be found that the three equations **39** (5) are each equivalent, when $\lambda = -9$, to

$$4l+m-n = 0,$$

thus verifying the lemma for the case of a double root of the discriminating cubic. This shows that the $X$-, $Y$-directions are any two orthogonal directions each orthogonal to $(4, 1, -1)$, which is therefore the $Z$-direction. For instance, the $X$-, $Y$-directions could be taken as $(0, 1, 1)$, $(1, -2, 2)$. (We are using d-r's, not d-c's.)

2. $18x^2+9y^2+14z^2-4yz+8zx+8xy-2x-6y-14z+6 = 0.$

The equations for the centre are

$$\begin{aligned} 18\xi+4\eta+4\zeta-1 &= 0,\\ 4\xi+9\eta-2\zeta-3 &= 0,\\ 4\xi-2\eta+14\zeta-7 &= 0, \end{aligned}$$

giving the unique solution $\xi = -3/14$, $\eta = 4/7$, $\zeta = 9/14$. Hence

$$d' = -\xi-3\eta-7\zeta+6 = 0.$$

The discriminating cubic is

$$\begin{vmatrix} 18-\lambda & 4 & 4 \\ 4 & 9-\lambda & -2 \\ 4 & -2 & 14-\lambda \end{vmatrix} = 0,$$

having roots 6, 14, 21. Since these are all positive, the quadric consists by **41** (v) of the *single point* $(\xi, \eta, \zeta)$.

3. $$x^2+2zx+2xy-2x+2y+2z-2 = 0.$$

Of the equations for the centre, two only are found to be independent giving the line of centres

$$\xi+1 = 0, \qquad \eta+\zeta-2 = 0.$$

Hence

$$d' = -\xi+\eta+\zeta-2 = 1.$$

The discriminating cubic is

$$\begin{array}{ccc} 1-\lambda & & \\ 1 & & \\ 1 & & -\lambda \end{array} = 0,$$

having roots $-1$, 2, 0. So the quadric is, in standard form,

$$X^2-2Y^2 = 1,$$

and is therefore by **41** (viii) a *hyperbolic cylinder*.

The principal direction corresponding to $\lambda = -1$ is, according to **39** (5), given by

$$l+m = 0, \qquad l+n = 0.$$

Thus the $X$-direction is $(1, -1, -1)$; similarly the $Y$-direction is $(2, 1, 1)$. These give the directions of the axes of any normal section. The $Z$-direction is then $(0, -1, 1)$, being that of the axis of the cylinder. The new origin is any point on this axis. (We are again using d-r's.)

4. $$2y^2-2yz-2zx+2xy-4x-3y+2z-1 = 0. \tag{1}$$

If we form the equations for the centre, we find they are inconsistent. (The reader should do this.) Hence the quadric is non-central. The discriminating cubic is

$$\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & 2-\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0,$$

having roots $3, -1, 0$.

We have

$$\Delta = \begin{vmatrix} . & 1 & -1 & -2 \\ 1 & 2 & -1 & -3/2 \\ -1 & -1 & . & 1 \\ -2 & -3/2 & 1 & -1 \end{vmatrix} = 9/4.$$

So by **42** (3) $w' = \pm\sqrt{(-9/4)/3(-1)} = \pm\sqrt{3}/2$. Therefore the quadric is, in standard form, by **42** (2),

$$3X^2-Y^2 = \pm\sqrt{3}Z, \tag{2}$$

and is, by **42** (xvi), a *hyperbolic paraboloid*.

The principal direction corresponding to $\lambda = 3$ is, according to **39** (5), given by

$$-3l+m-n = 0, \qquad l-m-n = 0.$$

Thus the $X$-direction is $(1, 2, -1)$; the $Y$-, $Z$-directions are similarly found to be $(1, 0, 1)$, $(1, -1, -1)$, expressed by d-r's.

This is an expeditious way of getting the form and orientation of the quadric, but not the position of the new origin.

The latter can be got as in **42**, remembering that the change of origin is there effected *after* the rotation of the axes, or as follows. Using the principal directions found, we write (*cf.* **5**)

$$\left.\begin{aligned} X = (x+2y-z+p)/\sqrt{6}, \qquad Y = (x+z+q)/\sqrt{2}, \\ Z = (x-y-z+r)/\sqrt{3}, \end{aligned}\right\} \tag{3}$$

and substitute in (2) and compare with (1). Since an unambiguous sign is taken for $Z$ in (3), the ambiguity in (2) is removed. We find that (1), (2) agree if the upper sign is taken and if $p = -2$, $q = 1$, $r = 5/2$. Thus the planes $X = 0$, $Y = 0$, $Z = 0$ are determined, and their common point, the new origin, is $(-1, 3/2, 0)$, giving the vertex of the paraboloid in the original coordinate-system.

5. $$9x^2+y^2+z^2-2yz+6zx-6xy+10x+8y-5 = 0. \tag{4}$$

We notice that the terms of the second degree form a perfect square $(3x-y+z)^2$ but that the l.h.s. is not reducible. Hence by **42** (xvii) the quadric is a *parabolic cylinder*; so we proceed to express (4) in the form

$$(3x-y+z+k)^2+2(ax+by+cz+d) = 0, \tag{5}$$

where the planes $3x-y+z = 0$, $ax+by+cz = 0$ are orthogonal, *i.e.*

$$3a-b+c = 0. \quad . \quad . \quad . \tag{6}$$

Comparing coefficients in (4), (5) and using (6), we find $k = 1$, $a = 2$, $b = 5$, $c = -1$, $d = -3$. Substituting in (5) and normalising, it becomes in standard form

$$Z^2 = -2(\sqrt{30}/11)Y,$$

where

$$Y = (2x+5y-z-3)/\sqrt{30}, \qquad Z = (3x-y+z+1)/\sqrt{11}.$$

Hence the parabolic cylinder has semi-latus rectum $\sqrt{30}/11$, and the line of vertices is the meet of the planes $Y = 0$, $Z = 0$, the former being the tangent plane along this line.

## 44. Properties derived from Standard Forms: Ellipsoid

Given a quadric of specified type, we may now suppose the axes so chosen that its equation takes the appropriate standard form amongst (i)–(xvii), and may use this form in the subsequent study of the quadric. The following work illustrates this; it is not intended as a catalogue of properties. For definiteness, properties of the sort discussed in the present section are given in the particular form applicable to the ellipsoid. But they hold good, with minor changes, for any non-singular central quadric, and the reader should write out the corresponding work for other cases. Alternatively, he should generalise it to apply to any such quadric by taking its equation in the form **46** (1).

**Conjugate diameters.** Consider the ellipsoid

$$x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2 = 1. \qquad (1)$$

Applying **40** (7) to this case, the diametral plane conjugate to the direction $(l, m, n)$ is

$$lx/\alpha^2+my/\beta^2+nz/\gamma^2 = 0. \qquad (2)$$

Hence, if $(l', m', n')$ is any direction in this plane, and consequently conjugate to the direction $(l, m, n)$, (2) gives $(l/\alpha)(l'/\alpha)+(m/\beta)(m'/\beta)+(n/\gamma)(n'/\gamma) = 0$, *i.e.* the directions $(l/\alpha, m/\beta, n/\gamma)$, $(l'/\alpha, m'/\beta, n'/\gamma)$ are orthogonal, and conversely. It follows that, *if* $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ *are the d-c's of a triad of conjugate diameters, then* $(l_1/\alpha, m_1/\beta, n_1/\gamma)$, $(l_2/\alpha, m_2/\beta, n_2/\gamma)$, $(l_3/\alpha, m_3/\beta, n_3/\gamma)$ *are d-r's of three mutually orthogonal directions, and conversely.* Let $(L_1, M_1, N_1)$, $(L_2, M_2, N_2)$, $(L_3, M_3, N_3)$ be the d-c's of the latter; these we call the *eccentric* directions corresponding to the diameters. It will be seen that we have the analogue of a well-known property of eccentric angles of conjugate diameters of an ellipse. We may now suppose the eccentric directions ordered so as to form a r.h set.

The point $A(\alpha L_1, \beta M_1, \gamma N_1)$ is an extremity of the diameter with direction $(l_1, m_1, n_1)$; for **OA** has the

required direction, and $A$ lies on the ellipsoid. We deduce:

(i) *The sum of the squares of three conjugate semi-diameters is constant and equal to* $\alpha^2+\beta^2+\gamma^2$. For the required sum is

$$\begin{aligned}(\alpha^2L_1^2+\beta^2M_1^2+\gamma^2N_1^2)+(\alpha^2L_2^2+\beta^2M_2^2+\gamma^2N_2^2)+(\alpha^2L_3^2+\beta^2M_3^2+\gamma^2N_3^2)\\ = \alpha^2(L_1^2+L_2^2+L_3^2)+\beta^2(M_1^2+M_2^2+M_3^2)+\gamma^2(N_1^2+N_2^2+N_3^2)\\ = \alpha^2+\beta^2+\gamma^2,\end{aligned}$$

using the property **5** (4) of orthogonal directions.

(ii) *The volume of the parallelepiped having three conjugate semi-diameters as adjacent edges is constant and equal to* $\alpha\beta\gamma$. For the volume is

$$\begin{vmatrix}\alpha L_1 & \beta M_1 & \gamma N_1\\ \alpha L_2 & \beta M_2 & \gamma N_2\\ \alpha L_3 & \beta M_3 & \gamma N_3\end{vmatrix} = \alpha\beta\gamma\begin{vmatrix}L_1 & M_1 & N_1\\ L_2 & M_2 & N_2\\ L_3 & M_3 & N_3\end{vmatrix} = \alpha\beta\gamma,$$

using the property **5** 4.

1. The ellipsoid referred to any triad of conjugate diameters as oblique axes is $x^2/R_1^2+y^2/R_2^2+z^2/R_3^2 = 1$, where $R_1$, $R_2$, $R_3$ are the lengths of the semi-diameters.

2. Let the space undergo a homogeneous strain so that lengths parallel to $OX$, $OY$, $OZ$ are multiplied by $1/\alpha$, $1/\beta$, $1/\gamma$, respectively. Then the ellipsoid becomes a unit sphere, while a triad of conjugate diameters becomes a triad of orthogonal diameters of the sphere having the corresponding eccentric directions, and conversely.

3. A pair of polar lines which meet at a point $P$ on the quadric are parallel to a pair of conjugate diameters each conjugate to $OP$.

**Principal axes of central section.** Consider the problem of finding the principal axes of the (elliptical) section of (1) by a given plane

$$\lambda x+\mu y+\nu z = 0. \quad . \quad . \quad . \quad (3)$$

We use the known property that these constitute the unique pair of diameters in the plane which are both

conjugate and orthogonal. Let their eccentric directions be $(L_1, M_1, N_1)$, $(L_2, M_2, N_2)$, and let $(L_3, M_3, N_3)$ be that of the diameter conjugate to both, *i.e.* conjugate to the plane (3). Since these directions are mutually orthogonal, by **5** 6,

$$\left.\begin{aligned} L_2 = M_3N_1 - M_1N_3, \qquad M_2 = N_3L_1 - N_1L_3, \\ N_2 = L_3M_1 - L_1M_3. \end{aligned}\right\} \qquad (4)$$

Since the *diameters* corresponding to $(L_1, M_1, N_1)$, $(L_2, M_2, N_2)$ are to be orthogonal, we have

$$\alpha^2L_1L_2 + \beta^2M_1M_2 + \gamma^2N_1N_2 = 0,$$

or, using (4) and rearranging,

$$(\beta^2-\gamma^2)M_1N_1L_3 + (\gamma^2-\alpha^2)N_1L_1M_3 + (\alpha^2-\beta^2)L_1M_1N_3 = 0. \quad (5)$$

Also $(L_1, M_1, N_1)$, $(L_3, M_3, N_3)$ are orthogonal, so that

$$L_1L_3 + M_1M_3 + N_1N_3 = 0. \quad . \quad . \quad (6)$$

Solving (5), (6) for $L_3$, $M_3$, $N_3$, we obtain

$$\frac{L_3}{L_1[N_1^2(\gamma^2-\alpha^2) - M_1^2(\alpha^2-\beta^2)]} = \frac{M_3}{M_1[L_1^2(\alpha^2-\beta^2) - N_1^2(\beta^2-\gamma^2)]} = \frac{N_3}{N_1[M_1^2(\beta^2-\gamma^2) - L_1^2(\gamma^2-\alpha^2)]}. \quad (7)$$

Now let $R_1$ be the length of the semi-diameter with direction $(l_1, m_1, n_1)$. Then

$$R_1^2 = \alpha^2L_1^2 + \beta^2M_1^2 + \gamma^2N_1^2,$$

whence

$$\begin{aligned} &N_1^2(\gamma^2-\alpha^2) - M_1^2(\alpha^2-\beta^2) \\ &\qquad = \alpha^2L_1^2 + \beta^2M_1^2 + \gamma^2N_1^2 - \alpha^2(L_1^2+M_1^2+N_1^2) = R_1^2 - \alpha^2, \end{aligned}$$

and so on.

Thus (7) may be written

$$L_1 : M_1 : N_1 = \frac{L_3}{R_1^2-\alpha^2} \cdot \frac{M_3}{R_1^2-\beta^2} \cdot \frac{N_3}{R_1^2-\gamma^2} \qquad (8)$$

Substituting these values in (6), we get the equation for $R_1$ as

$$L_3^2/(R_1^2-\alpha^2)+M_3^2/(R_1^2-\beta^2)+N_3^2/(R_1^2-\gamma^2) = 0. \qquad (9)$$

This is quadratic in $R_1$, for clearly the same equation must be satisfied by $R_2$, the other semi-axis. Finally, comparing (2), (3) and recalling the definition of $(L_3, M_3, N_3)$, we have

$$L_3 : M_3 : N_3 = \alpha\lambda : \beta\mu : \gamma\nu. \qquad (10)$$

Using (10), our results (8), (9) can now be restated: *If $R_1$, $R_2$ are the principal semi-axes of the section of* (1) *by the plane* (3), *then $R_1^2$, $R_2^2$ are the roots of*

$$\frac{\alpha^2\lambda^2}{R^2-\alpha^2}+\frac{\beta^2\mu^2}{R^2-\beta^2}+\frac{\gamma^2\nu^2}{R^2-\gamma^2} = 0, \qquad (11)$$

*and the d-r's of these axes are, when $R = R_1, R_2$, respectively,*

$$\left(\frac{\alpha^2\lambda}{R^2-\alpha^2}, \frac{\beta^2\mu}{R^2-\beta^2}, \frac{\gamma^2\nu}{R^2-\gamma^2}\right). \qquad (12)$$

Since all sections parallel to (3) are similarly situated ellipses, the latter directions are those of the principal axes of all such sections.

4. Recover these results thus: Let $(l, m, n)$ be the direction of the semi-diameter of the section having length $R$; then $(l^2/\alpha^2+m^2/\beta^2+n^2/\gamma^2)R^2 = 1$, $\lambda l+\mu m+\nu n = 0$. These give two sets of values of $l : m : n$, which coincide only if $R$ is a principal semi-axis. The condition for coincidence is (11), and when this is satisfied $l : m : n$ is given by (12).

**Circular sections.** The central section just considered is a circle if and only if $R_1 = R_2$, *i.e.* (11) has equal roots in $R^2$. Suppose $\alpha^2 > \beta^2 > \gamma^2$. Then, if $\mu \neq 0$, it is

easily seen that (11) has one root greater and one less than $\beta^2$. Hence it cannot have equal roots unless $\mu = 0$, in which case one root is $\beta^2$. The other root is also $\beta^2$ if and only if $\alpha^2(\beta^2-\gamma^2)\lambda^2+\gamma^2(\beta^2-\alpha^2)\nu^2 = 0$, *i.e.* (3) is either of the planes

$$\sqrt{\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)}x \pm \sqrt{\left(\frac{1}{\gamma^2}-\frac{1}{\beta^2}\right)}z = 0. \quad . \quad (13)$$

Therefore only planes parallel to these meet (1) in circular sections.

5. The centres of the circular sections lie on the diameters conjugate to the planes (13), *i.e.*

$$x/\alpha\sqrt{(\alpha^2-\beta^2)} = y/0 = \pm z/\gamma\sqrt{(\beta^2-\gamma^2)}.$$

These meet (1) in the points

$$\{\pm\alpha\sqrt{(\alpha^2-\beta^2)/(\alpha^2-\gamma^2)}, \quad 0, \quad \pm\gamma\sqrt{(\beta^2-\gamma^2)/(\alpha^2-\gamma^2)}\},$$

at which the tangent planes are parallel to (13). These points may be regarded as point-circles belonging to the systems of circular sections, and are called *umbilics*.

6. A necessary and sufficient condition for a quadric to possess (real) umbilics is that it should not possess (real) generators.

7. In the limit when $\gamma \to 0$ the ellipsoid becomes the disc bounded by the ellipse $x^2/\alpha^2+y^2/\beta^2 = 1$, $z = 0$, and the umbilics become the foci of this ellipse.

**Model of ellipsoid.** The properties of the circular sections can instructively be employed in making a model of an ellipsoid as follows; the reader is recommended to devote some little time to it. Draw the ellipse (fig. 5) which is to be the central section perpendicular to the circular sections. Insert two sets of symmetrically placed parallel chords as shown (about eight in each set) and number the intersections; these chords are to be the diameters, in this plane, of the circular sections. From thin cardboard cut circular discs of these diameters. Consider the disc (fig. 6) corresponding to $AA'$ in fig. 5. Draw its diameter $AA'$ and mark the points 1, . . ., 5; through these draw chords perpendicular to $AA'$.

Cut slits, of width about the thickness of the card, along half of each chord. Repeat for each disc, making the slits for

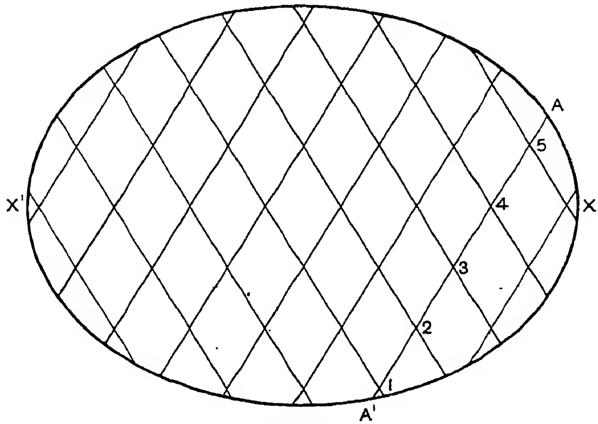


FIG. 5.

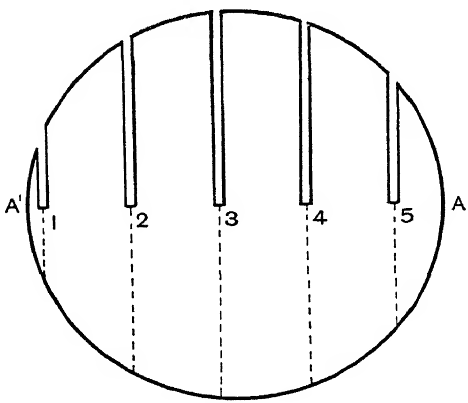


FIG. 6.

the two sets on opposite sides of the diameters, as viewed from, say, $X$. Fit the discs together so that the points bearing the same number on intersecting discs coincide. If necessary,

hinges made of gummed paper may be fixed along the intersections of the discs. A model having the general appearance of fig. 7 results.

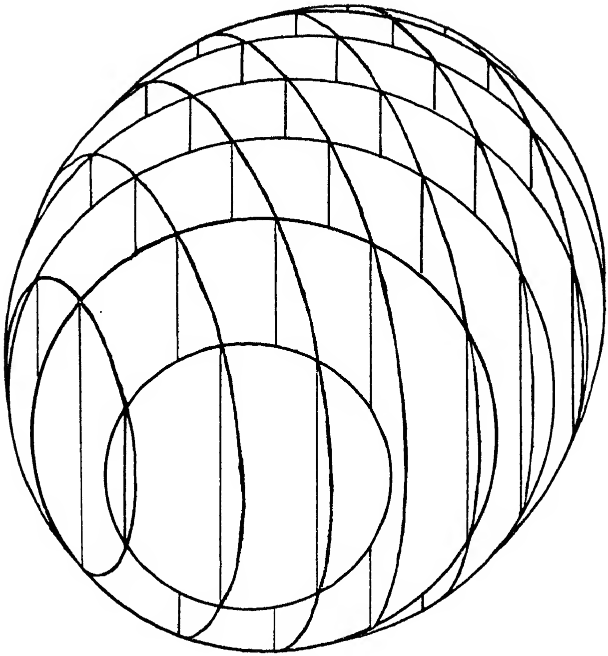

FIG. 7.

The model is *deformable* by varying the angle between the intersecting sections, and, in particular, it collapses into a plane in two ways. That it always retains the form of an ellipsoid is easily proved. For a simple discussion of collapsible models the reader is referred to H. W. Turnbull, *Edinburgh Math. Notes*, No. 32 (1941), pp. xvi–xix.

8. Ellipsoids with semi-axes $\alpha$, $\beta$, $\gamma$ and $\alpha'$, $\beta'$, $\gamma'$ can be deformed into one another in the manner described if

$$\beta = \beta' \quad \text{and} \quad \beta^2\{(\alpha^2-\alpha'^2)-(\gamma^2-\gamma'^2)\}-\alpha^2\gamma'^2+\alpha'^2\gamma^2 = 0.$$

## 45. Hyperboloid of One Sheet

**Generators.** Writing the equation **41** (ii) in the form

$$\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}=1-\frac{x^2}{\alpha^2} \quad \text{or} \quad \left(\frac{y}{\beta}-\frac{z}{\gamma}\right)\left(\frac{y}{\beta}+\frac{z}{\gamma}\right)=\left(1+\frac{x}{\alpha}\right)\left(1-\frac{x}{\alpha}\right) \tag{1}$$

we see that every point of each of the lines

$$\left.\begin{aligned}\frac{y}{\beta}+\frac{z}{\gamma}&=\lambda\left(1-\frac{x}{\alpha}\right)\\ \frac{y}{\beta}-\frac{z}{\gamma}&=\frac{1}{\lambda}\left(1+\frac{x}{\alpha}\right)\end{aligned}\right\} \quad (2) \qquad \begin{aligned}\frac{y}{\beta}-\frac{z}{\gamma}&=\mu\left(1-\frac{x}{\alpha}\right)\\ \frac{y}{\beta}+\frac{z}{\gamma}&=\frac{1}{\mu}\left(1+\frac{x}{\alpha}\right),\end{aligned} \quad (3)$$

lies on (1) for all values of the parameters $\lambda$, $\mu$. Hence, *as $\lambda$, $\mu$ vary,* (2), (3) *give the two reguli of generators of* (1). The $\lambda$-, $\mu$-generators intersect where

$$\left.\begin{aligned}x=\alpha(\lambda\mu-1)/(\lambda\mu+1), \qquad y=\beta(\lambda+\mu)/(\lambda\mu+1),\\ z=\gamma(\lambda-\mu)/(\lambda\mu+1),\end{aligned}\right\} \tag{4}$$

and thus we obtain a particular *rational parametric representation* of the surface (*cf.* **31** 4).

Each generator meets the principal elliptic section $x^2/\alpha^2+y^2/\beta^2=1$, $z=0$, in a single point; let this be $P(\alpha\cos\theta,\ \beta\sin\theta,\ 0)$. Any line $s$ through $P$ is given in parametric form by

$$x=\alpha\cos\theta+lr, \qquad y=\beta\sin\theta+mr, \qquad z=nr, \tag{5}$$

and so its points of intersection with the quadric correspond to the values of $r$ satisfying

$$(\alpha\cos\theta+lr)^2/\alpha^2+(\beta\sin\theta+mr)^2/\beta^2-n^2r^2/\gamma^2=1.$$

This gives either $r=0$, corresponding to $P$ itself, or

$$(l^2/\alpha^2+m^2/\beta^2-n^2/\gamma^2)r+2(l\cos\theta/\alpha+m\sin\theta/\beta)=0.$$

If $s$ is a generator, this must be satisfied for all $r$, *i.e.*

$$l^2/\alpha^2+m^2/\beta^2-n^2/\gamma^2=0, \qquad l\cos\theta/\alpha+m\sin\theta/\beta=0,$$

whence $l/\alpha \sin \theta = m/-\beta \cos \theta = n/\pm\gamma$. Substituting in (5), we obtain

$$\frac{x-\alpha \cos \theta}{\alpha \sin \theta} = \frac{y-\beta \sin \theta}{-\beta \cos \theta} = \frac{z}{\pm\gamma} \tag{6}$$

for the equations of the two generators through $P$. As $\theta$ varies from 0 to $2\pi$, (6) gives the two reguli.

1. (2), (3) are equivalent to (6) if $\lambda = \mu = \cot \theta/2$.

2. A non-rational parametric representation of (1) is $x = \alpha \cos \psi \sec \chi$, $y = \beta \sin \psi \sec \chi$, $z = \gamma \tan \chi$, the point $(\psi, \chi)$ so defined being the meet of two generators given by (6) when $\theta = \psi \pm \chi$, and when $+\gamma$, $-\gamma$, respectively, are taken in the last member.

3. The distances intercepted on the generators in 2 between $(\psi, \chi)$ and the plane $z = 0$ are

$$\{(\alpha^2+\gamma^2) \sin^2 (\psi \pm \chi)+(\beta^2+\gamma^2) \cos^2 (\psi \pm \chi)\}^{1/2} \tan \chi.$$

With given $\psi$, $\chi$ these distancs are constant if $\alpha^2+\gamma^2$, $\beta^2+\gamma^2$ are constant. If therefore in the family of hyperboloids $x^2/(\alpha^2+k)+y^2/(\beta^2+k)+z^2/(-\gamma^2+k) = 1$ $(k \leqslant \gamma^2)$, we let points having the same $\psi$, $\chi$ correspond, then the distances between corresponding points on corresponding generators are independent of $k$. It follows that, if a model of one of these surfaces be made with thin rigid rods, swivel-jointed at their intersections, as generators, it can be deformed into any other of the surfaces. [This was discovered by O. Henrici (1874) and models designed by him are preserved in the Royal College of Science.]

4. The generators through the point on the principal elliptic section with eccentric angle $\theta$ meet the section by the plane $z = \kappa$ in the points with eccentric angles $\theta \pm \chi$, where $\kappa = \gamma \tan \chi$.

**Model of hyperboloid.** For definiteness of description we consider the construction of a model of a central frustum of the surface

$$4x^2+9y^2-3z^2 = 4k^2 \quad . \quad . \quad . \tag{7}$$

showing twenty-four generators of each system.

The frustum is to be bounded by the elliptic sections in the planes $z = \pm 2k$. The latter can conveniently be provided by opposite sides of a cardboard box, $k$ being then determined by the size of the box. Draw on paper a circle (fig. 8) on a diameter $AA'$ of length $4k$. Starting from $A$, divide the circumference into 24 equal arcs and through the points of

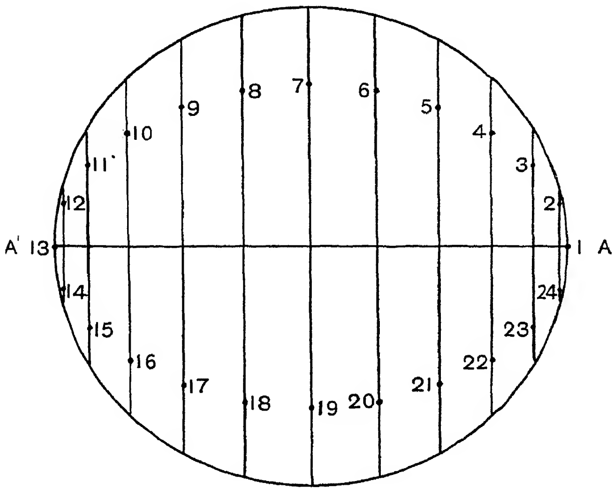


FIG. 8.

division draw chords perpendicular to $AA'$. On each chord mark the points whose distances from $AA'$ are one-third of its length; number these 1, 2, ..., 24 as shown. Trace these points on to the two sides of the box so that similarly numbered points are directly opposite each other, and pierce the card at every point. Lace a coloured silk thread through the holes so that it joins 1, 2, ..., 24 in one side to 9, 10, ..., 8 in the other; this shows one set of generators. Similarly, join 9, 10, ..., 8 in the first side to 1, 2, ..., 24 in the second by a differently coloured thread; this shows the other set of generators. Fig. 9 shows an oblique view of such a model (with one half the number of generators), thicker lines indicating generators in the front part.

Using results proved in the preceding examples, we verify that this construction does yield the required surface. For the elliptic section given by fig. 8 has principal semi-axes $2k$, $4k/3$, and so its equation referred to these axes is $4x^2+9y^2=16k^2$. As required, this is given by putting $z=\pm 2k$ in (7).

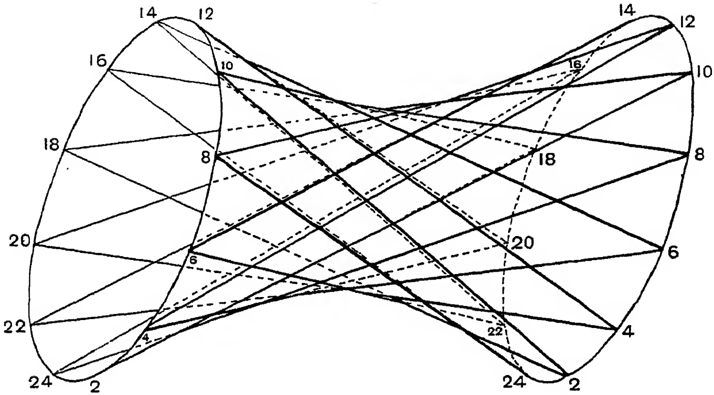


FIG. 9.

Moreover, the construction is such that the point marked $n$ has eccentric angle $(n-1)\pi/12$, and so the method of joining points of the two ellipses yields lines meeting them in points with eccentric angles of the form $\theta \pm \pi/3$. From 4, these lines are generators of the required type of surface. In the notation of 4 we have $\kappa=2k$, $\chi=\pi/3$, giving $\gamma=2k/\sqrt{3}$, in agreement with (7).

## 46. Central Quadric : Normals

Properties common to non-singular central quadrics can be studied by taking the equation of the surface $S$ as (*cf.* **41** (2))

$$ax^2+by^2+cz^2 = 1. \qquad (a,\, b,\, c \neq 0) \qquad (1)$$

The polar planes of $P'(x', y', z')$, $P''(x'', y'', z'')$ are

$$axx'+byy'+czz' = 1, \qquad axx''+byy''+czz'' = 1.$$

Their intersection is the polar line of $P'P''$, which consequently has d-r's $bc(y'z''-z'y'')$, etc. So *$P'P''$ is perpendicular to its polar line* if $\Sigma bc(x'-x'')(y'z''-z'y'') = 0$. Hence, if $P'$ is fixed, $P''$ must lie on the locus

$$\Sigma bc(x'-x)(y'z-z'y) = 0,$$

*i.e.*

$$\Sigma a(b-c)x'(y-y')(z-z') = 0, \quad . \quad (2)$$

which is a quadric cone with vertex $P'$.

**Normals.**—The normal at $P_1(x_1, y_1, z_1)$ on $S$ is the normal at $P_1$ to the tangent plane of $S$ at that point, *i.e.*

$$(x-x_1)/ax_1 = (y-y_1)/by_1 = (z-z_1)/cz_1 (= t, \text{ say}). \quad (3)$$

Let this contain the point $P'$. Then from (3)

$$x_1 = x'/(1+at), \quad y_1 = y'/(1+bt), \quad z_1 = z'/(1+ct), \quad (4)$$

and hence, since $(x_1, y_1, z_1)$ satisfies (1),

$$ax'^2/(1+at)^2+by'^2/(1+bt)^2+cz'^2/(1+ct)^2 = 1.$$

This is an equation of degree *six* for $t$; if any real root is substituted in (4) we get a point of $S$ the normal at which passes through $P'$. Hence *at most six* (real) *normals of $S$ pass through any given point*. Moreover, since the polar line of the normal at $P_1$ lies in the tangent plane at $P_1$, the normal is perpendicular to its polar line. Hence *the normals through $P'$ are included amongst the generators of the cone* (2).

Now let $P$, $Q$ be points of $S$ such that the normals at $P$, $Q$ intersect. Then the tangent planes at $PQ$ meet in a line perpendicular to $PQ$, *i.e.* $PQ$ is perpendicular to its polar line. Conversely, if the latter condition holds, the tangent planes at $P$, $Q$ meet in a line perpendicular to $PQ$, *i.e.* the normals at $P$, $Q$ intersect. If $P$ is fixed, the locus of $Q$ is therefore the intersection, say $\mathcal{K}$, of (1), (2) when $P' \equiv P$.

As $Q$ approaches $P$, $PQ$ becomes in the limit a tangent line $t$ of $\mathcal{K}$, and so also of $S$, at $P$. But, if a line is tangent to $S$ at $P$, so too is its polar line. Then the pair are by **44** 3 parallel to a pair of conjugate diameters of any section of $S$ parallel to the tangent plane at $P$. Moreover, since $PQ$ is perpendicular to its polar line, these conjugate diameters are orthogonal and are therefore the principal axes of the section. So $t$ can be parallel to either of these axes. Hence $P$ is, in general, a double point of $\mathcal{K}$ at which the tangents are parallel to the principal axes of any section of $S$ parallel to their plane. Their directions are called the *principal directions* at $P$. A curve * on $S$ whose direction at each point is a principal direction there is called † a *line of curvature.* There are thus two families of lines of curvature, one of each family in general passing through each point $P$ of $S$ and intersecting orthogonally the other line of curvature through $P$. A case of exception occurs when $P$ is an umbilic; for then the tangent plane at $P$ is parallel to circular sections of $S$, so that every pair of polar lines through $P$ is an orthogonal pair and the preceding discussion fails.‡ §

1. Writing $a, b, c = 1/A, 1/B, 1/C$, the condition **10** (5) shows that the normals at $P'(x', y', z')$, $P''(x'', y'', z'')$ on (1) intersect if and only if $\Sigma A(x'-x'')(y'z''-y''z') = 0$. Letting $(l, m, n)$, $(l_1, m_1, n_1)$ be the d-c's of $P'P''$, and of the normal to the plane $OP'P''$, this becomes

$$All_1+Bmm_1+Cnn_1 = 0.$$

But $ll_1+mm_1+nn_1 = 0$, and so for any values of $\kappa$, $\lambda$,

$$(\kappa A+\lambda)ll_1+(\kappa B+\lambda)mm_1+(\kappa C+\lambda)nn_1 = 0.$$

* This is *not* the curve $\mathcal{K}$.

† This is the single exception to our restriction of the term "line" to mean straight line.

‡ For the treatment of this case see, *e.g.*, Salmon, *Analytic Geometry of Three Dimensions*, § 301.

§ For a treatment of principal directions and lines of curvature from a different point of view see D. E. Rutherford, *Vector Methods* (in this series), § **24**.

Therefore, if the normals at the meets of any line with a central quadric intersect, so also do the normals at its meets with any confocal * ($\kappa = 1$), or with any similar coaxial quadric ($\lambda = 0$).

We shall see * that there are three confocals $S_1$, $S_2$, $S_3$ meeting orthogonally at any point $P$. Let a line $s$ through $P$ meet $S_1$, $S_2$, $S_3$ again in $Q_1$, $Q_2$, $Q_3$, and let the normals to $S_1$ at $P$, $Q_1$ intersect. Then the normals to $S_2$ at $P$, $Q_2$, and to $S_3$ at $P$, $Q_3$, also intersect. Clearly a possible choice of $s$ is the normal to $S_1$ at $P$; by the orthogonality property, this normal is tangent to $S_2$, $S_3$ and so to their curve of intersection. Therefore, when $s$ approaches this position, $Q_2$, $Q_3$ approach $P$, and so in the limit the direction of $s$ is a principal direction at $P$ on $S_2$, $S_3$. It follows that *the curve of intersection of two confocals is a line of curvature on each.*

2. Consider the extension of the work of this section to other types of quadric.

## 47. Cone

We shall take the general quadric cone **41** (vi) in the form

$$ax^2+by^2+cz^2 = 0. \qquad (1)$$

The d-c's $(l, m, n)$ of any generator then satisfy

$$al^2+bm^2+cn^2 = 0. \qquad (2)$$

**Orthogonal generators and tangent planes.** Suppose $(l, m, n)$ is perpendicular to a given direction $(l_1, m_1, n_1)$. Then eliminating $n$ between (2) and $ll_1+mm_1+nn_1 = 0$ gives

$$(an_1^2+cl_1^2)l^2+2cl_1m_1lm+(cm_1^2+bn_1^2)m^2 = 0. \qquad (3)$$

Let the roots of (3) correspond to values $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ of $(l, m, n)$; then from (3) and similar relations

$$l_2l_3/(cm_1^2+bn_1^2) = m_2m_3/(an_1^2+cl_1^2) = n_2n_3/(bl_1^2+am_1^2). \qquad (4)$$

Hence the condition $l_2l_3+m_2m_3+n_2n_3 = 0$ for these directions to be themselves orthogonal is

$$a(m_1^2+n_1^2)+b(n_1^2+l_1^2)+c(l_1^2+m_1^2) = 0. \qquad (5)$$

* Chapter VII.

If now the direction $(l_1, m_1, n_1)$ also is that of a generator, adding $al_1^2+bm_1^2+cn_1^2 = 0$ to (5) gives

$$a+b+c = 0. \quad . \quad . \quad . \quad (6)$$

Noticing the reversibility of the algebra, we see that (6) *is the necessary and sufficient condition for* (1) *to have a triad of mutually orthogonal generators.* This condition being independent of $(l_1, m_1, n_1)$, we see further that, if (1) possesses one such triad, then it possesses an infinite number.

1. The same result holds if the cone is given by

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0. \quad . \quad (7)$$

[(6) is an example of an *invariant* condition. Since intrinsic properties of a cone depend only on the roots of the discriminating cubic, such properties must be expressible in terms of the invariants **39** (10)–(12).]

2. The *reality* of the cone (1) and its orthogonal generators is ensured by (6). [(3) has real roots if $\Sigma bcl_1^2 \leqslant 0$. Using $\Sigma al_1^2 = 0$, $\Sigma a = 0$, this requires $\Sigma l_1^2(l_1^2-m_1^2)(l_1^2-n_1^2) \geqslant 0$, which is easily verified.]

The polar plane of $(x', y', z')$ w.r.t. (1) is

$$axx'+byy'+czz' = 0;$$

if $ax'/l = by'/m = cz'/n$, this is $lx+my+nz = 0$. Hence the latter contains its poles, and so is a tangent plane, if

$$l^2/a+m^2/b+n^2/c = 0, \quad . \quad . \quad (8)$$

giving the *tangential equation* of the cone.

The normal at the vertex $O$ to the tangent plane is $x/l = y/m = z/n$. Its locus, called the *reciprocal cone*, is therefore from (8)

$$x^2/a+y^2/b+z^2/c = 0. \quad . \quad . \quad (9)$$

The cone (1) has three mutually orthogonal tangent

planes if and only if the cone (9) has three mutually orthogonal generators, *i.e.*, by comparison with (6),

$$1/a+1/b+1/c = 0. \qquad (10)$$

3. The cone reciprocal to (7) is

$$\mathcal{A}x^2+\mathcal{B}y^2+\mathcal{C}z^2+2\mathcal{F}yz+2\mathcal{G}zx+2\mathcal{H}xy = 0.$$

Consequently (7) has three mutually orthogonal tangent planes if and only if

$$\mathcal{A}+\mathcal{B}+\mathcal{C} \equiv bc+ca+ab-f^2-g^2-h^2 = 0,$$

which expresses (10) in invariant form.

**Cone of revolution.** The cone (7), assumed real, is right circular if the discriminating cubic has a double root, $\lambda_1$ (say). By **39**, a necessary and sufficient condition for this, in the case $f, g, h \neq 0$, is $\mathcal{F}_{\lambda_1}, \mathcal{G}_{\lambda_1}, \mathcal{H}_{\lambda_1} = 0$. Eliminating $\lambda_1$ gives

$$\mathcal{F}/f = \mathcal{G}/g = \mathcal{H}/h. \qquad (11)$$

It is not immediately obvious that (11) is invariant. But it can be shown that the discriminant of $D_\lambda$, *i.e.* the condition for a double zero, is expressible as the sum of squares of functions of its coefficients **39** (10)–(12). The vanishing of this discriminant does in fact yield (11).

## 48. Hyperbolic Paraboloid

**Generators.** The equation **42** (xvi) may be written

$$(x/\alpha-y/\beta)(x/\alpha+y/\beta) = 2z/\gamma, \qquad (1)$$

whence, as in **45**, we see that its systems of generators are

$$\left.\begin{aligned} x/\alpha-y/\beta &= \lambda, \\ x/\alpha+y/\beta &= 2z/\lambda\gamma. \end{aligned}\right\} \quad (2) \qquad \left.\begin{aligned} x/\alpha+y/\beta &= \mu, \\ x/\alpha-y/\beta &= 2z/\mu\gamma. \end{aligned}\right\} \quad (3)$$

All the lines (2), (3) are parallel, respectively, to the fixed

planes $x/\alpha \mp y/\beta = 0$. Thence, or otherwise, the reader can verify the following constructions for models.

**Model of portion bounded by four generators.** Take a four-sided cardboard box, not necessarily rectangular but with opposite sides parallel. On the meets of these sides take four non-coplanar points $A$, $B$, $C$, $D$ (fig. 10). Join

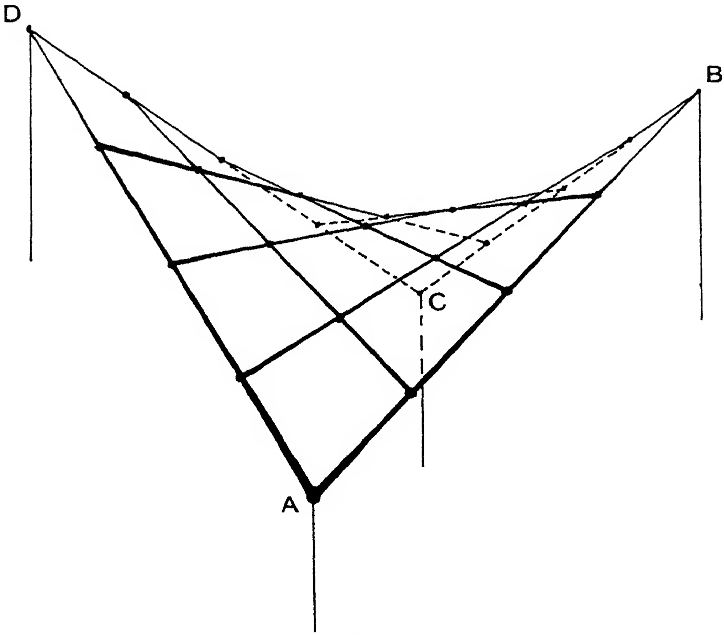


Fig. 10.

points of $AB$, $CD$ by threads parallel to the faces containing $AD$, $BC$; join points of $AD$, $BC$ by threads parallel to the faces containing $AB$, $CD$. These show generators of the unique paraboloid containing the skew quadrilateral $ABCD$. (In fig. 10 perspective is indicated by the varying thickness of the lines. The model should comprise more generators than are shown in the drawing.)

**Portion of surface near vertex.** Draw on cardboard the diagram in fig. 11 consisting of a rectangle $ABCD$, sides $2\alpha$, $2\beta$, with four symmetrically placed parabolic arcs, all of equal height $\frac{1}{2}\gamma$. Inscribe in $ABCD$ a series of parallelograms like $KLK'L'$; erect $KP$ perpendicular to $AB$ meeting the corresponding arc in $P$, and so on. Make holes where

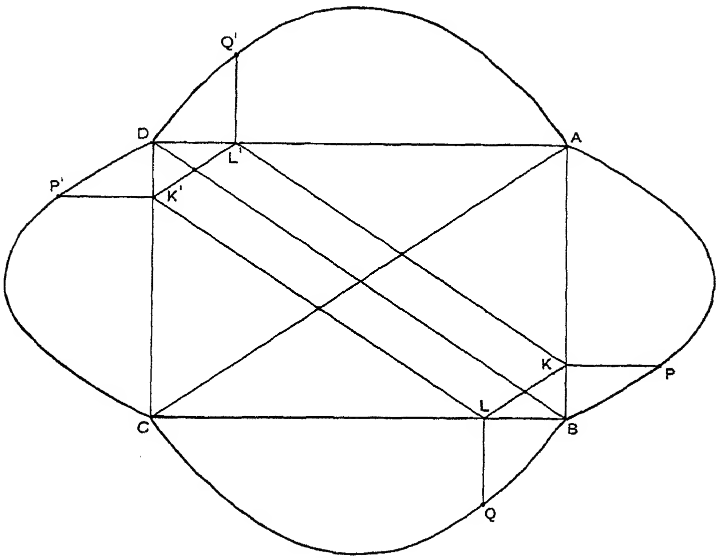


FIG. 11.

$KL$, $KL'$, etc. cross $BD$, $AC$. Cut round the outer boundary and fold along $AB$, etc., so that the planes of the parabolas become perpendicular to $ABCD$, the arcs on $AB$, $CD$ being on one side, those on $BC$, $DA$ on the other side, of $ABCD$. Stretch threads between $P$, $Q$; $P$, $Q'$, etc., passing through the appropriate holes on $BD$, $AC$. The result is such as that shown in fig. 12, the thicker lines indicating the front part. (The model should comprise about twice as many generators as appear in the figure.) It represents part of the surface

$x^2/\alpha^2 - y^2/\beta^2 = 2z/\gamma$, the tangent plane at the vertex being $ABCD$. [The reader may prefer to draw the boundary on the sides of a box, or to make it of wire.]

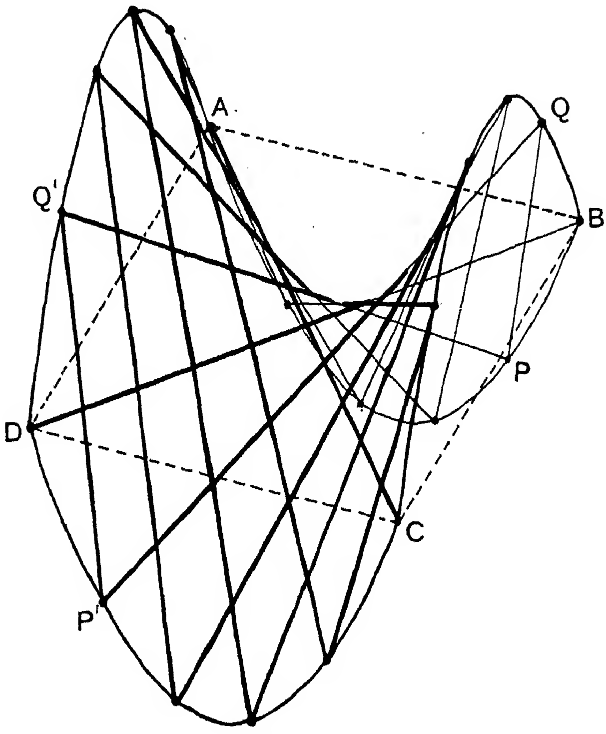


FIG. 12.

1. A stack of cards may be piled so that one set of corners lie above each other and the edges through them lie on paraboloids $zx = ky$, $zy = -kx$, the cards being parallel to the plane $z = 0$.

CHAPTER VII

# INTERSECTION OF QUADRICS: SYSTEMS OF QUADRICS

## 49. Space Quartic

Let $S'$, $S''$ be distinct quadrics in $\overline{\mathscr{E}}$; if they are reducible, let them have no common plane. Suppose there are points common to $S'$, $S''$; their aggregate $\mathscr{Q}$, say, consists of all points satisfying simultaneously

$$S' = 0, \qquad S'' = 0, \qquad . \qquad . \qquad (1)$$

which, taken together, are the equations of $\mathscr{Q}$.

Let $\Pi$, $\Lambda$ be fixed planes (whose meet lies in neither $S'$, $S''$) and consider the meets of $\mathscr{Q}$ with the plane $\varpi\Pi+\rho\Lambda$. These are given by

$$S' = 0, \qquad S'' = 0, \qquad \varpi\Pi+\rho\Lambda = 0, \qquad . \qquad (2)$$

which are three equations of degrees 2, 2, 1 for the three ratios * $x_1 : x_2 : x_3 : x_4$. By elementary algebra they have in general $2\times2\times1 = 4$ solutions, not necessarily real. So they give in general at most four points belonging to $\mathscr{Q}$ in the plane $\varpi\Pi+\rho\Lambda$. By suitably choosing $\varpi : \rho$ this plane can be made to contain any point of $\overline{\mathscr{E}}$, and so of $\mathscr{Q}$. Thus a variable point of $\mathscr{Q}$ is a function, at most 4-valued, of the single parameter $\varpi : \rho$. By **11**, $\mathscr{Q}$ is therefore a *curve*, as we naturally expect.

We define the *order* of a curve as the degree of the equation determining its intersections with an arbitrary

* Since the work applies to $\overline{\mathscr{E}}$, it is supposed that homogeneous coordinates are employed.

plane, whether or not the roots are all real. Hence $\mathcal{Q}$ is of order 4, *i.e.* a *quartic.* A quartic given by the meet of two quadrics is said to be of the *first species*; there is a second species of algebraic quartic not so given.* In what follows, "quartic" will mean an algebraic quartic of the first species.

By a familiar argument, every quadric $S$ whose equation can be put in the form $S \equiv \lambda' S' + \lambda'' S'' = 0$, where $\lambda'$, $\lambda''$ are any numbers, contains $\mathcal{Q}$. So $\mathcal{Q}$ is determined by any two distinct quadrics of this system.

Now let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(8)}$ be eight given points. Substituting their coordinates in the equation $a_{rs}x_r x_s = 0$ of the general quadric we get eight homogeneous linear equations for the ten coefficients $a_{rs}$. These equations have, in general, rank 8 and their solution is of the form $a_{rs} = \lambda' a'_{rs} + \lambda'' a''_{rs}$, where $a'_{rs}$, $a''_{rs}$ $(r,\ s = 1,\ \ldots,\ 4)$ are two linearly independent sets of constants, and $\lambda'$, $\lambda''$ are arbitrary. Hence every quadric through the given points is of the form

$$S \equiv \lambda' S' + \lambda'' S'' = 0,$$

where

$$S' \equiv a'_{rs}x_r x_s, \qquad S'' \equiv a''_{rs}x_r x_s,$$

and every quadric of this form passes through those points. But we have just seen that every such $S$ passes through a fixed quartic (provided $S'$, $S''$ have no common plane, which is the case if no six of the given points are coplanar). Hence *there is a quartic $\mathcal{Q}$ through any eight given points and $\mathcal{Q}$ is, in general, unique.*

1. All the quadrics through *seven* given points may in general be written $S \equiv \lambda' S' + \lambda'' S'' + \lambda''' S''' = 0$, where $S'$, $S''$, $S'''$ are fixed quadrics. These all contain every point satisfying simultaneously $S' = S'' = S''' = 0$. These equations have eight solutions, of which seven are the given points. Hence

* Salmon, *Analytic Geometry of Three Dimensions*, § 347. It must be pointed out that only a bare introduction to these topics is within the scope of this text. A fairly full account from an elementary standpoint is given by Salmon.

*every quartic through seven given points passes also through an eighth fixed point.*

2. $\mathcal{Q}$ meets a quadric, which does not contain it, in at most eight (real) points.

If an arbitrary plane $\Theta$ meets $S'$, $S''$ it does so in conics $\Gamma'$, $\Gamma''$, say. These in general meet in 0, 2, or 4 (real) points giving the points of $\mathcal{Q}$ in $\Theta$, and so we verify the previous result that there are in general at most four points of $\mathcal{Q}$ in any plane. We now see that the only exceptions occur:

(i) If $\Gamma'$, $\Gamma''$ coincide ($\equiv \Gamma$, say). We can take $\lambda' : \lambda''$ so that $S \equiv \lambda' S' + \lambda'' S''$ contains any given point; let it contain a point of $\Theta$ not on $\Gamma$. By **30** 4, $S$ then contains $\Theta$, and the rest of $S$ (if any) is another plane $\Phi$. If $\Phi$ meets $S'$ it does so in a conic $\Delta$, say. But $\mathcal{Q}$ is the meet of $S$, $S'$, and hence in this case $\mathcal{Q}$ reduces to the two conics $\Gamma$, $\Delta$.

(ii) If $\Gamma'$, $\Gamma''$ reduce to line-pairs $s$, $s'$ and $s$, $s''$ having one line $s$ in common. Then $S'$, $S''$ have the common generator $s$, and this supplies one of the solutions of (2) for all $\varpi : \rho$. The rest of $\mathcal{Q}$ accounts for the other three solutions and so is a curve of order 3, *i.e.* a *cubic* $\mathcal{C}$. *We suppose* $\mathcal{C}$ *not to reduce further*, for, if it does, we get back to case (i).

3. If in (i) $\Gamma$, $\Delta$ meet, they do so in one or two points, or else coincide; if $S'$, $S''$ are not reducible, they touch at every common point of $\Gamma$, $\Delta$.

4. $\mathcal{Q}$, $\mathcal{C}$ are not plane curves. [A plane cannot meet a quadric in a quartic or cubic.]

## 50. Space Cubic

A space cubic $\mathcal{C}$ is not the *complete* intersection of any two algebraic surfaces. For the intersection of surfaces of degrees $m$, $n$ has order $mn$ (by the argument applied to equations **49** (2)); so it has order 3 only if $m = 1$, $n = 3$, or *vice versa*. But then one of the surfaces is a plane and the intersection is a plane curve.

Thus the simplest possibility is for $\mathcal{C}$ to be the residual

intersection of two surfaces which have also merely a line in common. This is precisely how $\mathcal{C}$ has arisen here, the surfaces being quadrics. Every algebraic space cubic can be so given (see Salmon, *op. cit.*, § 333).

Let, then, $\mathcal{C}$ be given as in **49** (ii). The lines $s'$, $s''$ meet in a point $M$, say, giving the only point in $\Theta$ not on $s$ which is common to $S'$, $S''$, and $M$ is a point of $\mathcal{C}$. Thus, in every plane $\Phi$ containing $s$ there is, in general, a unique point $P$ of $\mathcal{C}$ not on $s$; conversely, through every point $P$ of $\mathcal{C}$ not on $s$ there is a unique plane $\Phi$ containing $s$. If $\Phi_1$, $\Phi_2$ are any two planes through $s$, and if we write $\Phi \equiv \lambda_1\Phi_1+\lambda_2\Phi_2$, then to every value of $t = \lambda_1 : \lambda_2$ there corresponds a unique plane $\Phi$ containing $s$, and to every plane $\Phi$ containing $s$ there corresponds a unique value of $t$. Thus we can establish a (1–1) correspondence between the points $P$ and the values of the parameter $t$. Moreover the equations determining $P$ are algebraic. Therefore the coordinates of $P$ are one-valued algebraic functions of $t$, *i.e.* rational functions of $t$. Hence, by using a suitable common denominator, the homogeneous coordinates $x_r$ of $P$ are proportional to polynomials in $t$. Substituting these in the equation $\xi_r x_r = 0$ of an arbitrary plane $\Pi$, we get an algebraic equation for the values of $t$ corresponding to the points in which $\mathcal{C}$ meets $\Pi$. But, by the definition of a cubic curve, this equation must have degree 3. Hence the coordinates of $P$ must be expressible in the form $x_r(t)$, $(r = 1, \ldots, 4)$, where $x_r(t)$ is, in general, a polynomial of degree 3 in $t$.

Now let $S \equiv a_{rs}x_r x_s = 0$ be an arbitrary quadric. Substituting $x_r = x_r(t)$, we get an equation of degree at most 6 in $t$. Its roots, when substituted back in $x_r(t)$, give the coordinates of the points of intersection of $\mathcal{C}$, $S$. But there is certainly a quadric $S$ (not unique) through any seven given points; take these to be points of $\mathcal{C}$. Then the equation of degree $\leqslant 6$ is satisfied by *seven* values of $t$, and so by every value. Thus a *quadric which meets $\mathcal{C}$ in seven points entirely contains $\mathcal{C}$.*

Take any seven points $P_1, P_2, \ldots, P_7$ of $\mathcal{C}$, and let $Q$ be any other point on $P_1P_2$. By **49** there is a "pencil" * of quadrics $S \equiv \lambda' S' + \lambda'' S''$ through these eight points. By the preceding result, every $S$ contains $\mathcal{C}$. Also the line $P_1P_2$ meets every $S$ in three points $P_1$, $P_2$, $Q$ and so is a generator $s$ of $S$. Hence the quartic $\mathcal{Q}$ determined by $S'$, $S''$ consists of $\mathcal{C}$ and $s$. Thus *there is a pencil of quadrics which contain $\mathcal{C}$ and any fixed chord of $\mathcal{C}$.*

1. In general, a plane meets $\mathcal{C}$ in one or three (real) points. [The cubic equation for the intersections has real coefficients, and so has at least one real root; the other two roots are both real or both not real.]

2. If the plane $\Phi$ meets $\mathcal{C}$ elsewhere than at $P$ it does so in two points on $s$. [$P$ is the only intersection not on $s$; if there are other intersections they must be on $s$, and by 1 there must be two of them.]

3. $\mathcal{C}$ has no trisecants and no double points; a tangent $p$ at a point $P$ on $\mathcal{C}$ meets $\mathcal{C}$ in no other point; any plane through $p$ meets $\mathcal{C}$ in one and only one other point; two bisecants of $\mathcal{C}$ meet either on $\mathcal{C}$ or not at all. [Observe that the contrary of any of these would imply that the equation determining the intersections of $\mathcal{C}$ with some plane has four roots; since $\mathcal{C}$ is a cubic, this would mean that $\mathcal{C}$ lies entirely in that plane, contradicting **49** 4.]

4. All the quadrics $S$ of the above pencil touch at $P_1$, $P_2$. [$\mathcal{C}$ lies on $S$; so the tangent line $p_1$ of $\mathcal{C}$ at $P_1$ is a tangent line of $S$; $s$ is a generator of $S$ through $P_1$; since it meets $\mathcal{C}$ again at $P_2$, it is, by 3, distinct from $p_1$. Therefore the plane of $p_1$, $s$ is the tangent plane at $P_1$ of every $S$ (unless $P_1$ is a singularity of $S$).]

Now let $R$ be any other point on $P_1P_3$; we can choose $\lambda' : \lambda''$ so that $S$ contains $R$. Then $S$ contains the line $P_1P_3$ giving a second generator $r$ of $S$ through $P_1$. By 3, 4, $s$, $r$, $p_1$ are not coplanar. Hence three non-coplanar tangent lines of $S$ go through $P_1$, which is therefore a singularity of $S$. Also $S$ cannot have any other singularity;

* See **51**.

otherwise it would be reducible, but we know that it contains $\mathcal{C}$, which is not a plane curve. Hence $S$ is a quadric cone with vertex $P_1$, and the join of every point of $\mathcal{C}$, being a point of $S$, to $P_1$ is a generator of $S$. Moreover, every generator $g$ of $S$ meets $\mathcal{C}$; for the plane of $g$, $s$ meets $\mathcal{C}$ in $P_1$, $P_2$ on $s$; therefore, by 1, it meets $\mathcal{C}$ in a third point $P$; by 3, $P$ is not on $s$; but $P$ must be on $S$; therefore $P$ is on $g$. Thus *the chords of $\mathcal{C}$ through any fixed point of $\mathcal{C}$ generate a quadric cone.* Consequently, *$\mathcal{C}$ is determined by the intersection of any two such cones, the rest of the intersection being the join of their vertices.* Now, if $P_1, \ldots, P_6$ are any six points of $\mathcal{C}$, there is a unique quadric cone with vertex $P_1$ containing $P_2, \ldots, P_6$; for the joins of $P_1$ to $P_2, \ldots, P_6$ meet an arbitrary plane in five distinct points (distinct, on account of 3); there is a unique conic through these five points, and this is a section of the required cone. Similarly, there is a unique quadric cone with vertex $P_6$ containing $P_1, \ldots, P_5$. The intersection of these cones determines $\mathcal{C}$ uniquely. It follows that *is completely determined when any six of its points are iven.*

## 51. Pencil of Quadrics

$S' \equiv a'_{rs}x_rx_s = 0$, $S'' \equiv a''_{rs}x_rx_s = 0$ being distinct quadrics, the family $(S)$

$$S(\lambda', \lambda'') \equiv \lambda'S' + \lambda''S'' = 0 \qquad (\lambda', \lambda'' \text{ any numbers}) \quad (1)$$

is called a *pencil of quadrics.** There is a unique quadric $S$ corresponding to each value of $\lambda' : \lambda''$. If $S'$, $S''$ intersect, they define a quartic $\mathcal{Q}$ and every $S$ contains $\mathcal{Q}$. The following properties are true in general, but they are subject to exceptions which we have not space to enumerate.

The coefficients of $S$ are linear in $\lambda'$, $\lambda''$. Hence

* $S'$ or $S''$ need not contain any points, *e.g.* $S'$, $S''$ may be positive definite forms; then there are no points satisfying $S'=0$ or $S''=0$ but there are points satisfying (1) if $\lambda'$, $\lambda''$ have opposite signs.

*a unique* $S$ *goes through any given point* (not on $\mathcal{Q}$, if this exists). *The polar plane of a fixed point* $\mathbf{y}$ w.r.t. $S$, being

$$\lambda' a'_{rs} x_r y_s + \lambda'' a''_{rs} x_r y_s = 0, \quad . \quad . \quad (2)$$

*contains in general the fixed line* $a'_{rs} x_r y_s = 0$, $a''_{rs} x_r y_s = 0$, for all $\lambda'$, $\lambda''$. This is called the *conjugate line* of $\mathbf{y}$ w.r.t. $(S)$. However, the plane (2) is independent of $\lambda' : \lambda''$ if $\mathbf{y}$ is such that there exists a number $\mu$ giving

$$a'_{rs} y_s = \mu a''_{rs} y_s. \quad (r = 1, \ldots, 4) \qquad (3)$$

(3) have a solution if and only if the determinant

$$| a'_{rs} - \mu a''_{rs} | = 0. \quad . \quad . \quad . \quad (4)$$

This gives in general four values of $\mu$, not necessarily real.* If they are real and distinct, substitution in (3) gives four points $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(4)}$ each having the same polar plane w.r.t. every $S$; no other point has this property. But the meet of the polar planes of $\mathbf{y}^{(2)}$, $\mathbf{y}^{(3)}$, $\mathbf{y}^{(4)}$, being the pole of their plane w.r.t. every $S$, has this property and is therefore $\mathbf{y}^{(1)}$, and so on. Hence $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(4)}$ *are the vertices of a unique tetrahedron self-polar w.r.t. every* $S$.

The condition for $S$ to be singular is

$$| \lambda' a'_{rs} + \lambda'' a''_{rs} | = 0;$$

this is (4), if $\lambda'/\lambda'' = -\mu$, and (3) are then the equations for the singularity. Hence, *corresponding to each point of the self-polar tetrad there is a cone belonging to* $(S)$ *and having that point as vertex.*

The coefficients of the tangential equation (**37** (5)) of $S$ are of degree three in those of the point equation (1), and so are cubic forms in $\lambda'$, $\lambda''$. Hence, by **37** 2, the pole of a fixed plane $\Pi$ w.r.t. $S$ are cubic forms in $\lambda'$, $\lambda''$. Therefore, as $\lambda' : \lambda''$ varies, *the pole generates a space cubic* $\mathcal{C}$ called the *polar cubic* of $\Pi$. In general, $\mathcal{C}$ meets $\Pi$ in three points, each corresponding to a quadric of the pencil with respect to which the pole of $\Pi$ lies in $\Pi$, and which therefore touches $\Pi$. The pole of $\Pi$ w.r.t. any cone being

* Turnbull and Aitken, *Canonical Matrices* (1932), 108, Ex. 6.

the vertex (see **35**), $\mathcal{C}$ contains the points of the self-polar tetrad. If $\Pi \equiv \Omega$, then $\mathcal{C}$ becomes the *centre-locus*.

As an instance of the application of these properties, let $S_1$ be a given quadric and $O$ a given point. Take any sphere $S_0$, centre $O$, and consider the pencil $\lambda_0 S_0 + \lambda_1 S_1 = 0$; let $\mathcal{C}$ be its centre-locus. $\mathcal{C}$ meets $S_1$ in at most six points (see **50**); let $P$ be one of these. The polar plane of $P$ w.r.t. $S_1$ is $T$, the tangent plane at $P$. But $P$ is the centre of some quadric $S$ of the pencil, so the polar plane of $P$ w.r.t. $S$ is $\Omega$. Let $T$, $\Omega$ meet in $t$; then $t$ is the conjugate line of $P$. Therefore the polar plane of $P$ w.r.t. $S_0$ contains $t$ and so is parallel to $T$. But this plane is normal to $OP$. Therefore $OP$ is normal to $S_1$ at $P$. Conversely, if $OP$ is normal to $S_1$ at $P$, then $P$ is on $\mathcal{C}$. Hence *at most six normals of $S_1$ contain $O$*. Since $O$ is on $\mathcal{C}$, the chords of $\mathcal{C}$ through $O$ generate a quadric cone. Hence *the normals from $O$ to $S_1$ lie on a quadric cone which contains the centre of $S_1$* (*cf.* **46**).

1. The coefficients of the line-equation of $S$ (see **33** (4)) are quadratic in those of the point-equation; hence at most two quadrics of the pencil touch a given line.

2. The sections of $(S)$ by any plane $\Pi$ form a pencil of conics $(\Gamma)$. If $\mathcal{Q}$ exists and if $\Pi$ meets $\mathcal{Q}$ in four points, every $\Gamma$ contains these points; $(\Gamma)$ then includes three line-pairs, being the sections of three quadrics $S$ touching $\Pi$.

3. $(S)$ includes one or three paraboloids.

4. If $\mathbf{y}$ varies on a fixed line $s$, its conjugate line generates a quadric, the *polar quadric* of $s$ w.r.t. $(S)$. [Take $\mathbf{y} = \mu\mathbf{z} + \nu\mathbf{t}$, ($\mathbf{z}$, $\mathbf{t}$ fixed).] The polar lines of $s$ w.r.t. $(S)$ are the other generators of this quadric.

5. Consider some cases of exception in regard to the self-polar tetrahedron, *e.g.* where $\mathcal{Q}$ reduces to a conic-pair or a skew quadrilateral.

## 52. Range of Quadrics

Let $\Sigma'$, $\Sigma''$ be distinct quadrics having *tangential* equations $\Sigma' = 0$, $\Sigma'' = 0$. Then the family of quadrics $(\Sigma)$ given by

$$\Sigma(\mu', \mu'') \equiv \mu'\Sigma' + \mu''\Sigma'' = 0$$

is called a *range of quadrics*. The properties of a range are therefore dual to those of a pencil, and the reader should write them out.

## 53. Confocal Quadrics

Consider a given central quadric *

$$S \equiv x^2/A + y^2/B + z^2/C - 1 = 0. \qquad (A > B > C) \quad (1)$$

Let $P(\xi, \eta, \zeta)$ be any point,

$$\Pi \equiv l(x-\xi) + m(y-\eta) + n(z-\zeta) = 0$$

a plane through $P$, and $Q$ the pole of $\Pi$ w.r.t. $S$. We propose first to find if it is possible to choose $\Pi$ so that $PQ$ is normal to $\Pi$.

The coordinates of $Q$ are easily found to be

$$(Al, Bm, Cn)/(\xi l + \eta m + \zeta n).$$

Hence d-r's of $PQ$ are

$$(\xi^2 - A)l + \xi\eta m + \xi\zeta n, \qquad \eta\xi l + (\eta^2 - B)m + \eta\zeta n,$$
$$\zeta\xi l + \zeta\eta m + (\zeta^2 - C)n.$$

Therefore $PQ$ is normal to $\Pi$ if there exists a number $k$ such that

$$\left.\begin{aligned} (\xi^2 - A)l + \xi\eta m + \xi\zeta n &= kl, \\ \eta\xi l + (\eta^2 - B)m + \eta\zeta n &= km, \\ \zeta\xi l + \zeta\eta m + (\zeta^2 - C)n &= kn. \end{aligned}\right\} \quad . \qquad . \quad (2)$$

We consequently encounter an application of **39** II. The "discriminating cubic" is here

$$D_k \equiv \begin{vmatrix} \xi^2 - A - k & \xi\eta & \xi\zeta \\ \eta\xi & \eta^2 - B - k & \eta\zeta \\ \zeta\xi & \zeta\eta & \zeta^2 - C - k \end{vmatrix}$$
$$\equiv -(k+A)(k+B)(k+C) + \Sigma\xi^2(k+B)(k+C) = 0. \qquad (3)$$

Assuming $\xi, \eta, \zeta \neq 0$ and putting $k = \infty, -C, -B, -A$ in $D_k$, it has signs $-, +, -, +$. Hence (3) has roots $k_1, k_2, k_3$

* We now revert to rectangular cartesian coordinates.

such that $k_1 > -C > k_2 > -B > k_3 > -A$. By **39** II, their substitution in (2) gives a unique triad of mutually orthogonal directions corresponding, say, to positions $\Pi_1, \Pi_2, \Pi_3$ and $Q_1, Q_2, Q_3$ of $\Pi, Q$. Since $PQ_1$ is normal to $\Pi_1$ and $PQ_2, PQ_3$ are orthogonal to $PQ_1$, $\Pi_1$ contains $Q_2, Q_3$, and so on. Hence $\Pi_1, \Pi_2, \Pi_3$ are *a unique triad of orthogonal planes through $P$ conjugate w.r.t. $S$.*

Now if $D_k$ has a double zero, giving, say, $k_1 = k_2 \neq k_3$, then by **39** II, $\Pi_3$ has a unique direction, while $\Pi_1, \Pi_2$ can have any directions orthogonal to $\Pi_3$ and to each other. If this happens, there is a single infinity of triads of orthogonal planes through $P$ conjugate w.r.t. $S$. We then call $P$ a *focus* * of $S$. But we have just seen that, if $\xi, \eta, \zeta \neq 0$, $D_k$ has no repeated zero. Suppose then (say) $\zeta = 0$, so that

$$D_k \equiv \{-(k+A)(k+B)+\xi^2(k+B)+\eta^2(k+A)\}(k+C).$$

The first factor is seen not to have a double zero; hence $D_k$ has a double zero if and only if $k = -C$ is a zero of the first factor, *i.e.* if

$$\frac{\xi^2}{A-C}+\frac{\eta^2}{B-C} = 1. \quad . \quad . \quad (4)$$

Hence every point of the conic (4) lying in the plane $\zeta = 0$ is a focus; this is called a *focal conic* of $S$. The other focal conics are similarly

$$\eta = 0, \quad \frac{\zeta^2}{C-B}+\frac{\xi^2}{A-B} = 1, \quad . \quad (5)$$

$$\xi = 0, \quad \frac{\eta^2}{B-A}+\frac{\zeta^2}{C-A} = 1. \quad . \quad (6)$$

However, $B-A$, $C-A < 0$, so (6) is not real; (4) is an ellipse and (5) a hyperbola.

1. One and only one of the focal conics intersects $S$; it does so in four points.

* Compare the analogous definition of a focus of a conic, *e.g.* Filon, *Projective Geometry* (1935), § 115.

2. If $P$ is outside $S$, then $\Pi_1, \Pi_2, \Pi_3$ are the principal planes of the tangent cone from $P$.

3. Discarding the assumption $A > B > C$, when $\xi, \eta, \zeta \neq 0$, $D_k$ has a repeated zero if and only if $A = B = C$. Then $S$ is a sphere, and every point is a focus of $S$.

4. By **39** I, $D_k$ has a triple zero if and only if

$$\eta\zeta = \zeta\xi = \xi\eta = 0, \qquad \xi^2 - A = \eta^2 - B = \zeta^2 - C.$$

This requires $\eta = \zeta = 0$, $B = C$, $\xi^2 = A - B$, or analogous relations given by cyclic permutation. In this case every orthogonal triad through $P$ is conjugate; $P$ is called a *principal focus* of $S$. It follows that $S$ has principal foci if and only if it is a quadric of revolution; they are then the foci of the meridian section on the axis of symmetry.

Two quadrics $S$, $S'$ are said to be *confocal* if they have the same foci and so the same focal conics. This requires $S$, $S'$ to have the same principal planes. Hence if $S$ is given by (1), $S'$ must be of the form

$$S' \equiv x^2/A' + y^2/B' + z^2/C' - 1 = 0.$$

Then, by (4)–(6), $S$, $S'$ have the same focal conics if and only if

$$B - C = B' - C', \qquad C - A = C' - A', \qquad A - B = A' - B'.$$

These are equivalent to $A - A' = B - B' = C - C' = \lambda$ (say). Hence all the quadrics confocal to $S$ are given by

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1. \quad (\lambda \text{ arbitrary}) \qquad (7)$$

This is called a *confocal system*.

Taking an arbitrary plane in the form

$$lx + my + nz + p = 0,$$

the tangential equation of (7) is

$$(A+\lambda)l^2 + (B+\lambda)m^2 + (C+\lambda)n^2 - p^2 = 0,$$

*i.e.*

$$Al^2 + Bm^2 + Cn^2 - p^2 + \lambda(l^2 + m^2 + n^2) = 0. \qquad (8)$$

Hence this is a particular instance of a *range* of quadrics. Moreover, if $\Sigma = 0$ is the tangential equation of $S$ in any rectangular system, since $l^2+m^2+n^2$ is invariant (being interpretable as the square of the length of a vector), the confocal system is simply

$$\Sigma+\lambda(l^2+m^2+n^2) = 0. \quad . \quad . \quad (9)$$

5. The last result can be shown to hold whether or not $S$ is central. Deduce that the system confocal to the paraboloid $x^2/A+y^2/B = 2z/c$ is

$$\frac{x^2}{A+\lambda}+\frac{y^2}{B+\lambda} = \frac{2z}{c}+\frac{\lambda}{c^2}.$$

If the quadric (7) contains a given point $P(\xi, \eta, \zeta)$, we have

$$(\lambda+A)(\lambda+B)(\lambda+C)-\Sigma\xi^2(\lambda+B)(\lambda+C) = 0. \quad (10)$$

This is just (3), with $k = \lambda$, and so has the roots $k_1$, $k_2$, $k_3$ with the properties already derived, which may be written, when $\xi, \eta, \zeta \neq 0$,

$$\left.\begin{array}{lll} A+k_1 > 0, & B+k_1 > 0, & C+k_1 > 0; \\ A+k_2 > 0, & B+k_2 > 0, & C+k_2 < 0; \\ A+k_3 > 0, & B+k_3 < 0, & C+k_3 < 0. \end{array}\right\} \quad . \quad (11)$$

Therefore *through any point $P$ there pass in general three quadrics of the system,* these being from (11) an ellipsoid, a hyperboloid of one sheet, a hyperboloid of two sheets. Moreover, solving (2) with $k = k_1, k_2, k_3$, we find

$$l : m : n = \xi/(A+k_1) : \eta/(B+k_1) : \zeta/(C+k_1); \text{ etc.} \quad (12)$$

Hence $\Pi_1$ is the plane through $(\xi, \eta, \zeta)$ with normal having d-r's given by (12), and this is seen from (7) to be the tangent plane of (7) at $(\xi, \eta, \zeta)$ when $\lambda = k_1$. Thus *the planes $\Pi_1$, $\Pi_2$, $\Pi_3$ are the tangent planes of the confocals through $P$; therefore these confocals meet orthogonally at $P$.*

Further, the definition of $Q_1$, $Q_2$, $Q_3$ shows that $Q_1Q_2$ is the polar line of $PQ_3$. Now let $P$ be on $S$, so that (say)

$k_1 = 0$, and $\Pi_1$ is the tangent plane of $S$ at $P$. Then $Q_1$ coincides with $P$, and so $PQ_2$, $PQ_3$ become orthogonal polar lines in $\Pi_1$. By **44** 3, they therefore have the directions of the principal axes of any section of $S$ parallel to $\Pi_1$. Putting then $\lambda = k_1 = 0$ in (10), we get

$$ABC - \Sigma\xi^2 BC = 0, \qquad (13)$$

whence

$$\left.\begin{aligned} ABC\Sigma(A-\xi^2) &= \Sigma\xi^2 BC(B+C), \\ ABC\Sigma[(BC-\xi^2(B+C))] &= \Sigma\xi^2 B^2C^2. \end{aligned}\right\} \qquad (14)$$

Subtracting (13) from (10) and dividing by $\lambda$, we obtain as the equation for $k_2$, $k_3$,

$$\lambda^2 + \lambda\Sigma(A-\xi^2) + \Sigma(BC-\xi^2(B+C)) = 0.$$

Multiplying by $ABC$ and using (13), (14) this becomes

$$\lambda^2\Sigma\xi^2 BC + \lambda\Sigma\xi^2 BC(B+C) + \Sigma\xi^2 B^2C^2 = 0,$$

which may be written in the form

$$\xi^2/A(\lambda+A) + \eta^2/B(\lambda+B) + \zeta^2/C(\lambda+C) = 0. \qquad (15)$$

Using (12) with $k_1 = 0$, and writing $\lambda = -R^2$, (15) becomes

$$Al^2/(R^2-A) + Bm^2/(R^2-B) + Cn^2/(R^2-C) = 0. \qquad (16)$$

Now this is equivalent to **44** (11) applied to $S$, showing that $-k_2$, $-k_3$ are the squares of the semi-axes of the central section of $S$ parallel to $\Pi_1$.

Combining these results and restating them for the case $k_1 \neq 0$, we have therefore proved: *If $S_1$, $S_2$, $S_3$ with parameters $k_1$, $k_2$, $k_3$ are confocals meeting in $P$ and if $\Pi_1$ is the tangent plane of $S_1$ at $P$, then the principal semi-axes of the central section of $S_1$ parallel to $\Pi_1$ have the directions of the normals at $P$ to $S_2$, $S_3$ and their squares are $k_1-k_2$, $k_1-k_3$.*

Finally, if $P$ is on $S$ and is also a focus of $S$, the orthogonal polar lines $PQ_2$, $PQ_3$ become indeterminate. This means that the sections parallel to $\Pi_1$ are circles. Hence

*the points in which a focal conic meets $S$ are umbilics on $S$* (see 1). It is seen that the converse is also true, *i.e.* every umbilic is on a focal conic. Thus the properties of circular sections could be recovered from the present work.

6. If $S_1$, $S_2$ are intersecting confocals, and if radii of $S_1$ are drawn parallel to the normals of $S_2$ along the meet of $S_1$, $S_2$, these radii have constant length.

7. The locus of the poles of a given plane $\Pi$ w.r.t. a confocal system is a line normal to $\Pi$; one and only one confocal touches $\Pi$.

8. If $k_1$, $k_2$, $k_3$ are the parameters of the confocals through $P(\xi, \eta, \zeta)$, then

$$\begin{aligned}\xi^2 &= (k_1+A)(k_2+A)(k_3+A)/(B-A)(C-A),\\ \eta^2 &= (k_1+B)(k_2+B)(k_3+B)/(C-B)(A-B),\\ \zeta^2 &= (k_1+C)(k_2+C)(k_3+C)/(A-C)(B-C).\end{aligned}$$

Thus $k_1$, $k_2$, $k_3$ constitute a new set of (curvilinear) coordinates in $\mathcal{E}$. [Put $D_k \equiv -(k-k_1)(k-k_2)(k-k_3)$ and take $k = -A, -B, -C$ in $D_k$ as given by (3).]

# NOTE ON ABSTRACT GEOMETRY

In this book we have been content to start with an assumed knowledge of the rudiments of geometry such as most of us possess at this stage in our studies. This knowledge is usually a mixture of deductions from euclidean axioms and of intuitive notions. For instance, we used in **3** the result that there are two senses for the displacement of a point along a line. Would the reader care to say if this follows logically from his axioms, and, if not, to say just what other assumptions have been introduced?

Now our goal in geometry is to construct an abstract deductive system, *i.e.* to exhibit the logical consequences of an explicitly stated set of postulates. The result is a purely mental creation. There can be any number of abstract geometries, and, moreover, equivalent geometries may be derivable from various sets of postulates.

More precisely, a geometry G is a collection of propositions having the following properties:

(i) If a finite set $I$ of the propositions are selected as "initial propositions" the remaining propositions can be deduced from them.

(ii) The propositions $I$ must be logically consistent.

(iii) The propositions $I$ should be independent, *i.e.* none can be deduced from the others. (This is a natural, though not essential, requirement.)

It is desirable that $I$ should be simple in content. They are the postulates mentioned above and there is no question of proving them. Also there is no question of defining the entities or the relations involved in them. We start, in fact, with indefinable elements which are postulated to satisfy certain indefinable relations.

If $I'$ is any subset of $I$ then $I'$ must certainly satisfy conditions (ii), (iii). All the propositions which can be deduced from $I'$ alone constitute a geometry G$'$, and G$'$ is included in G. (Here a statement that a set of elements *may or may not* satisfy a condition is not regarded as a proposition.) Now the selection of $I$ is arbitrary, subject to conditions (i)–(iii). But it is natural in constructing G to select $I$ in such a manner that it consists of subsets $I_1, I_2, \ldots, I_n$ which render the geometry $\mathrm{G}_r$ derived from $I_1 + \ldots + I_r$ $(r=1, \ldots, n)$ an "interesting" geometry. $\mathrm{G}(\equiv \mathrm{G}_n)$ is then developed through the interesting stages $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_n$.

In the case of the real euclidean geometry of three dimensions with which this book is concerned two selections of postulates (apart from minor variations) have become traditional. One leads more immediately to the euclidean expression of results, while the other reaches them through stages that are interesting in their relation to different geometries. Here we have opportunity merely to allude to these stages.

We first state a set of initial propositions of *incidence* such as "Two distinct points determine one and only one line on which they both lie," and add propositions which ensure that the number of points is infinite and that the geometry shall be three dimensional. This yields *pure projective geometry*. We then proceed to express the work in an algebraic symbolism, and obtain what is sometimes called *analytical projective geometry*, though it is merely a translation into symbols of pure projective geometry. Coordinates are used, but it is not until initial propositions of *order* and *continuity* are introduced that we can treat the coordinates as real numbers. When this is done we have *real projective geometry*, which is formally identical with the geometry of $\bar{\mathcal{E}}$ in this text, but is got without using metrical notions. However, no fresh initial propositions are needed for its metrical formulation. Any quadric $Q$ in the space is singled out and the relation of any pair of points to $Q$ is

used to define the distance between them, and the relation of any pair of intersecting lines to $Q$ to define the angle between them. If $Q$ is allowed to degenerate in a particular manner into a single plane, the "plane at infinity," distance and angle so defined acquire their euclidean character and the geometry expressed in terms of them becomes *real euclidean geometry*.

There is an apparently easier method of getting formally the same result. In its simplest version it is to *define* a point as an ordered set of three real numbers, space as the set of all such points, a plane as the set of points whose coordinates satisfy a single linear relation, and a line as the set of points whose coordinates satisfy two such relations. Thence we can deduce the properties of parallelism. We then *define* the distance between two points by the formula **2** (1), and the angle between two intersecting lines by **4** (3). The rest of the work is developed in formally much the same way as in this book (with or without using "points at infinity," according to taste).

The outcome is then not itself a geometry in the sense described above. It contains no initial propositions and therefore no indefinables. Instead, it starts merely from definitions and for the rest consists entirely of theorems deduced from them with the aid of mathematical analysis. It becomes in fact a branch of such analysis. But analysis has its own initial propositions and, taken as a whole, does constitute a geometry in the previous sense. This includes real euclidean geometry, though partly expressed in a different mathematical language.

For a simple discussion of these matters the reader is referred to G. H. Hardy, "What is Geometry?" *Math. Gazette*, **12** (1926), 309–316; for a fuller but not too technical account, to J. W. Young, *Projective Geometry*, Carus Math. Monographs, No. 4 (1930); and for a concise technical account, to G. de B. Robinson, *Foundations of Geometry* (Toronto, 1940). The last supplies adequate further references.

As regards further reading on the subject-matter of the rest of the book the following suggestions are offered:

For a slightly more elementary approach and for examples:

W. H. Macaulay, *Solid Geometry* (Cambridge, 1930).

S. L. Green, *Algebraic Solid Geometry* (Cambridge, 1941).

For more advanced work:

G. Salmon, *Analytic Geometry of Three Dimensions*, I (7th ed., Longmans, 1928).

D. M. Y. Sommerville, *Analytical Geometry of Three Dimensions* (Cambridge, 1934).

# SUBJECT INDEX